Strong Mobility in Weakly Disordered Systems

E. Ben-Naim[1] and P. L. Krapivsky[2]

[1] *Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*

[2] *Department of Physics, Boston University, Boston, Massachusetts 02215*

We study transport of interacting particles in weakly disordered media. Our one-dimensional system includes (i) disorder: the hopping rate governing the movement of a particle between two neighboring lattice sites is inhomogeneous, and (ii) hard core interaction: the maximum occupancy at each site is one particle. We find that over a substantial regime, the root-mean-square displacement of a particle, σ, grows super-diffusively with time t, $\sigma \sim (\epsilon\, t)^{2/3}$, where ϵ is the disorder strength. Without disorder the particle displacement is sub-diffusive, $\sigma \sim t^{1/4}$, and therefore disorder strongly enhances particle mobility. We explain this effect using scaling arguments, and verify the theoretical predictions through numerical simulations. Also, the simulations show that regardless of disorder strength, disorder leads to stronger mobility over an intermediate time regime.

PACS numbers: 02.50.-r, 05.40.-a, 78.55.Qr, 66.10.cg

Disorder, inhomogeneities, and impurities are ubiquitous in physical systems and are widely used to control properties of matter. Some of the most fascinating phenomena in contemporary physics including localization [1–3], glassiness [4, 5], slow relaxation [6], and frustration [7] are unique consequences of disorder.

While the effects of disorder on noninteracting particles are well-understood, the influence of disorder on interacting, strongly correlated particles remain an open question [8]. In a quantum system, an isolated particle is localized by disorder, but localization can be destroyed when there are two interacting particles [9]. Hence, disorder and particle interactions compete. We investigate this competition between disorder and interaction in a classical system where inhomogeneities are known to trap particles and drastically decrease their mobility. We find that disorder leads to super-diffusive displacements over an intermediate regime whereas in the absence of disorder, the displacements are sub-diffusive.

We generalize the standard exclusion process [10] to study the interplay between disorder and interactions [11–13]. Our system is an infinite one-dimensional lattice whose sites may be either occupied by a single particle or vacant. Initially, the lattice is populated at random by identical particles with concentration c. Each particle may hop from an occupied site into a neighboring vacant site and this diffusion process is governed by the following rates: $p_+(i)$ is the hopping rate from site i to site $i+1$, and similarly, $p_-(i)$ is the hopping rate from site i to site $i-1$. While the total hopping rate is uniform, and is set to one without loss of generality, $p_-(i)+p_+(i)=1$, the lattice is inhomogeneous. At every site there is, with equal probabilities, a bias to the right, $p_+ = 1/2+\epsilon$, or a bias to the left, $p_+ = 1/2-\epsilon$, as illustrated in figure 1. The parameter $0 \le \epsilon \le 1/2$ is the disorder strength. Note that the disorder is quenched, uncorrelated, and uniform in strength. Moreover, since every lattice site accommodates a single particle, the particles interact via hard core repulsion. Our problem generalizes two well-known processes: single-file diffusion [14–18] with interaction but no disorder corresponds to the limit $\epsilon \to 0$, and Sinai diffusion [19–23] with disorder but no interaction corresponds to the limit $c \to 0$.

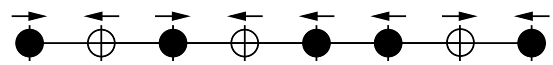

FIG. 1: Illustration of the disordered interacting particle system. The arrows indicate the bias at each site, the circles indicate vacant sites, and the bullets indicate occupied sites.

Our focus is transport in this disordered interacting particle system. Since there is no overall bias in either direction, on average, the displacement of a particle with respect to its initial position, x, does not change with time, $\langle x\rangle = 0$. The brackets denote an average over all realizations of the random process and over all disorders. We ask the most elementary question: how does the root-mean-square displacement, σ, defined by $\sigma^2 = \langle x^2\rangle$, evolve? We address this question via a scaling analysis of weakly disordered systems, $\epsilon \ll 1$, and numerical simulations with general disorder strengths.

Early Times. When disorder is weak, $\epsilon \ll 1$, there is an initial period during which particles do not "feel" the disorder and hence move at random, $p_+ = p_- = 1/2$. In this early regime, disorder is irrelevant and the behavior is dominated by particle interactions. Without disorder, the hard core repulsion causes a dramatic change in mobility: whereas an isolated particle moves diffusively, $\sigma \sim t^{1/2}$, the root-mean-square displacement of an interacting particle grows sub-diffusively with time [14, 15]

$$\sigma \sim t^{1/4}. \tag{1}$$

Therefore, the movement of a particle is severely hindered by the presence of other particles. We illustrate this remarkable collective behavior for extremely dense systems [15] where there are large clusters of occupied sites that are separated by isolated vacancies. Particles move by exchanging their position with neighboring vacancies. Furthermore, the sparse vacancies can be regarded as non-interacting [15]. A particle that, up to time t, exchanges position with a total of $N = N_+ + N_-$ vacancies of which N_+ were initially located to its right and N_- were initially located to its left, has the displacement $x = N_+ - N_-$. First, since the vacancies are randomly distributed in the initial configuration, the excess of vacancies in one direction follows from the law of large numbers, $|N_+ - N_-| \sim N^{1/2}$, and consequently, $x \sim N^{1/2}$. Second, vacancies that were initially located at a distance on the order of the diffusive length scale $t^{1/2}$ from a particle may exchange position with it. Hence $N \sim (1-c)\, t^{1/2}$ and combining this scaling law with $x \sim N^{1/2}$ yields (1). Although this scaling argument applies to densely packed systems, the behavior (1) holds for arbitrary concentrations [14, 15]. This suppressed diffusion is a direct consequence of the hard core interaction.

Intermediate Times. Eventually, the disorder becomes relevant, and the biased hopping rates do affect the particle displacement. Although there is no global bias in the hopping rates, there certainly are local biases, as illustrated in figure 1 where sites with negative bias are in the majority. We expect that, at least at intermediate times scales, or equivalently, intermediate length scales, these local biases lead to directed motion [24].

To quantify how such local biases affect particle mobility, we consider a particle that visits σ distinct sites of which n_+ have a positive bias and n_- have a negative bias with $\sigma = n_+ + n_-$. Since the disorder is uncorrelated, the difference between the number of positive and negative sites, $\Delta = |n_+ - n_-|$, grows diffusively with the total number of visited sites, $\Delta \sim \sigma^{1/2}$. The excess of sites biased in one direction leads to a drift in this preferred direction with the small velocity $v \sim \epsilon\, \Delta/\sigma$ or $v \sim \epsilon\, \sigma^{-1/2}$. Furthermore, the ballistic length scale $x \sim v\, t$ gives an estimate for the displacement, $x \sim (\epsilon\, t)\, \sigma^{-1/2}$. Since the displacement must be of the same order as the total number of sites visited, $x \sim \sigma$, we have

$$\sigma \sim \epsilon\, t\, \sigma^{-1/2}. \tag{2}$$

We thus arrive at our main result: the displacement becomes super-diffusive because of the disorder,

$$\sigma \sim (\epsilon\, t)^{2/3}. \tag{3}$$

Of course, this length scale ultimately exceeds the suppressed diffusion length scale (1). Hence, the inhomogeneous hopping rates generate a stochastic local velocity field, and as a result, there are local drifts that significantly enhance the mobility of the particles.

Late Times. To understand the behavior at late times, we recall that the displacement of a non-interacting particle in a random disorder is logarithmically slow [19–21]

$$x \sim \epsilon^{-2} (\ln t)^2. \tag{4}$$

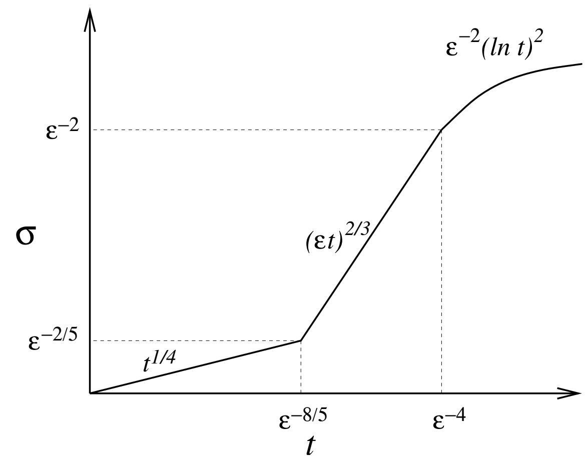

FIG. 2: The three regimes of behavior (5). The displacement σ is plotted versus time t using a double logarithmic scale.

At sufficiently large length scales, the random disorder generates a potential well that confines the particle. The depth of this potential well is the sum of all the biases in a given range, $U(x) = \sum_{i=1}^{x} (p_+(i) - p_-(i))$, and therefore, the depth of the well grows diffusively with distance, $U \sim \epsilon\sqrt{x}$. This stochastic well constitutes a barrier that the particle must overcome, and since the time to escape out of this barrier grows exponentially with the depth of the well, $t \sim \exp(U) \sim \exp(\epsilon\sqrt{x}\,)$, the displacement is logarithmic as in (4).

We argue that the slow mobility (4) also characterizes the asymptotic late time behavior of interacting particles in disorder. First, the confining potential well remains the same even when there are multiple particles. Second, the probability that a given particle escapes the well is exponentially small, and therefore, only mildly affected by the presence of other particles. We envision a scenario where particles are stuck in a local minimum of the potential and escape the barrier one at a time. Such an escape process is dominated by the same exponential escape time that characterizes an isolated, non-interacting particle. Therefore at late times, interacting particles in a random disorder also follow the logarithmic displacement law (4). Particle interactions become irrelevant and the behavior is governed by disorder alone.

Three Time Regimes. By combining the early (1), intermediate (3), and late (4) time behaviors, we conclude that the mobility of a given particle exhibits three distinct regimes of behavior (see also figure 2):

$$\sigma \sim \begin{cases} t^{1/4} & t \ll \epsilon^{-8/5}, \\ (\epsilon\, t)^{2/3} & \epsilon^{-8/5} \ll t \ll \epsilon^{-4}, \\ \epsilon^{-2} (\ln t)^2 & \epsilon^{-4} \ll t. \end{cases} \tag{5}$$

The time and length scales that characterize the crossover points can be obtained by matching the two corresponding behaviors. The transition from the early regime into

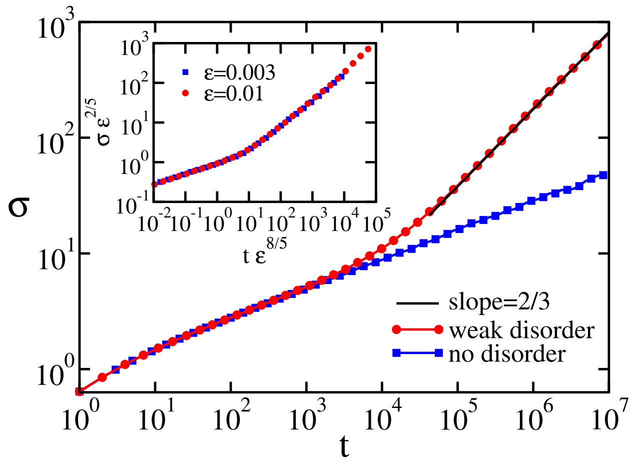

FIG. 3: The early and intermediate behaviors for weak disorder, $\epsilon = 10^{-2}$ (bullets) and no disorder, $\epsilon = 0$ (squares). Shown is the displacement σ versus time t as well as a reference line with slope 2/3. The inset shows that the scaled displacement $\sigma\epsilon^{2/5}$ is a universal function of the scaled time $t\epsilon^{8/5}$ using $\epsilon = 0.01$ (bullets) and $\epsilon = 0.003$ (squares).

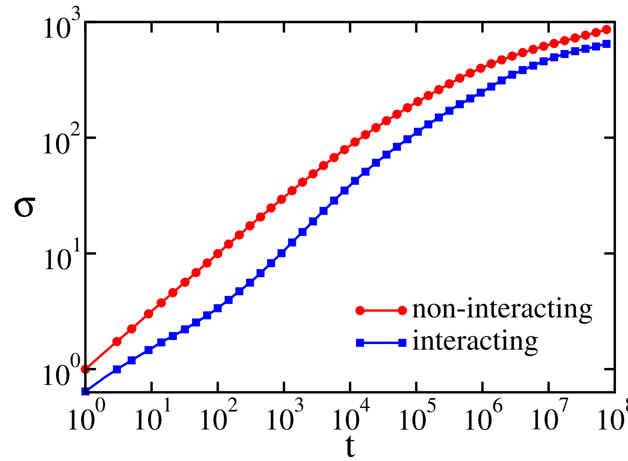

FIG. 4: The late time behavior for $\epsilon = 10^{-1}$. Shown is the displacement σ versus time t for non-interacting particles (bullets) and for interacting particles (squares).

the intermediate regime occurs at time $t \sim \epsilon^{-8/5}$ and length $\sigma \sim \epsilon^{-2/5}$, while the transition from the intermediate domain into the late domain occurs at time $t \sim \epsilon^{-4}$ [25] and length $\sigma \sim \epsilon^{-2}$, as shown in figure 2. Since this displacement length scale diverges as the disorder strength diminishes, the mobility enhancement effect becomes stronger as the disorder weakens.

Let us recap the three regimes of behavior. At the early stages, particle interactions dominate over disorder, and the motion of particles is sub-diffusive due to the hard core repulsion. In the intermediate regime, disorder and interactions are both relevant. The particles stream following the stochastic local velocity field and the result is a strong, super-diffusive transport. At late times, disorder dominates and interactions become irrelevant. Particles are trapped by a stochastic potential well and the displacement is logarithmically slow because the escape time is exponentially large.

As further support of the scaling behavior above, we can show that the stochastic potential well plays no role in the intermediate regime. Clearly, since the overall hopping rate equals one, the time scale characterizing the movement between neighboring sites is also of order one. The time to escape out of a well grows exponentially with the depth of the well, $t \sim \exp(U)$, but this time scale becomes appreciable only when the depth of the potential well is large, $\epsilon\sqrt{x} \gg 1$, or equivalently, when the displacement becomes sufficiently large, $x \gg \epsilon^{-2}$. Indeed, this length scale is realized only at the late time regime, as shown in figure 2. Therefore, trapping is negligible throughout the intermediate regime.

Let us now consider the effect of disorder on a non-interacting particle. In the absence of disorder, $\epsilon = 0$, the particle displacement is unhindered and thus, purely diffusive, $\sigma \sim t^{1/2}$. In weak disorder, $\epsilon \ll 1$, an isolated particle undergoes ordinary diffusion at early times, but is later slowed down considerably according to (4). Hence, there are two distinct regimes of behavior when interactions are absent [26]

$$\sigma \sim \begin{cases} t^{1/2} & t \ll \epsilon^{-4}, \\ \epsilon^{-2}(\ln t)^2 & \epsilon^{-4} \ll t. \end{cases} \tag{6}$$

We note that the crossover time scale matches the upper time scale in (5). Thus, in the absence of particle interactions, disorder slows the particles down.

Numerical Simulations. We performed extensive Monte Carlo simulations to test the scaling predictions. The simulations are a straightforward implementation of the transport process. Initially, identical particles randomly occupy the sites of a one-dimensional lattice of size L with periodic boundary conditions, and the initial concentration equals c. Each lattice site has a bias in the positive or the negative direction as $p_+ = 1/2 + \epsilon$ or $p_- = 1/2 - \epsilon$ with equal probabilities. The dynamics are asynchronous. In an elementary step, a randomly chosen particle hops to the right with probability p_+ or to the left with probability p_-, and this hop is successful only if the neighboring site is vacant. Subsequently, time is augmented by the inverse number of particles, $t \to t + 1/N$. This elementary step is repeated indefinitely. We present results of simulations with a system large enough to emulate an infinite lattice, $L = 4 \times 10^5$, and $c = 1/2$.

We verified the super-diffusive behavior (3) using a weak disorder (figure 3). Even though the super-diffusive behavior is an intermediate asymptotic, the duration of this regime grows rapidly as the disorder weakens. We checked that: (i) as implied by (5), a universal function describes how the scaled displacement $\sigma\epsilon^{2/5}$ depends on the scaled time $t\epsilon^{8/5}$ (figure 3, inset), (ii)

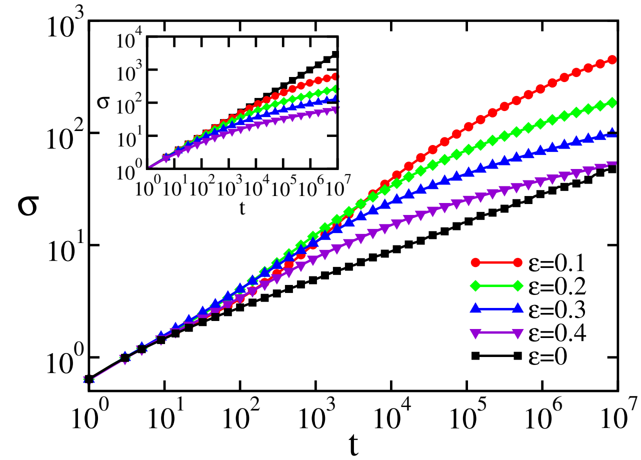

FIG. 5: The displacement σ versus time t for interacting particles (main figure) and noninteracting particles (inset) without disorder ($\epsilon = 0$, squares) and with moderate disorder values of $\epsilon = 0.1$ (bullets), $\epsilon = 0.2$ (diamonds), $\epsilon = 0.3$ (down-triangle), and $\epsilon = 0.4$ (up-triangle).

the concentration does not play an important role using $c = 1/4$ and $c = 3/4$, and (iii) a different type of disorder where p_+ is drawn from a flat distribution in the range $[1/2 - \epsilon : 1/2 + \epsilon]$ gives qualitatively similar results.

To test the behavior at late times, we also simulated a non-interacting particle system by ignoring the site occupancy restriction. These simulations show that after an extremely long transient period, the displacements of interacting particles and non-interacting particles nearly match (figure 4), thereby confirming that hard core interactions become irrelevant asymptotically, and that the behavior is governed by disorder alone.

Our scaling analysis tacitly assumes that disorder is small. A comparison of the behaviors with moderate disorders and with no disorder shows that regardless of the disorder strength, mobility is strengthened by disorder over a substantial regime (figure 5). Thus, mobility enhancement does not necessarily require weak disorder.

Figure 5 also shows that the displacement in a homogeneous system eventually catches up with the displacement in a strongly inhomogeneous system. Indeed, the sub-diffusive behavior (1) that characterizes a uniform system eventually exceeds the logarithmic displacement (4) in a disordered system. However, the crossover time $t \sim \epsilon^{-8}$ is very large at weak disorders and in practice, disorder enhances mobility over a substantial time regime. Moreover, the simulations show that the crossover time is large even at moderate and strong disorders (figure 5). Finally, as suggested by (6), disorder slows down noninteracting particles (figure 5, inset).

In conclusion, we studied how disorder affects transport in an interacting particle system. We found that the displacement is super-diffusive over a substantial period in a disordered system whereas the displacement is sub-diffusive without disorder. Therefore, there is an intricate interplay between interaction and disorder.

Disorder provides a mechanism for controlling transport properties because weak disorders result in a prolonged enhancement of mobility. This effect can be tested experimentally in colloidal [16] or biological [17, 18] channels, where the slow transport (1) was realized.

We thank Nigel Goldenfeld, Sidney Redner, and Stuart Trugman for useful discussions. We are grateful for financial support from DOE grant DE-AC52-06NA25396, NSF grants CHE-0532969 and CCF-0829541.

[1] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[2] D. J. Thouless, Physics Reports **67**, 5 (1980).
[3] E. Abrahams, P. W. Anderson, D. C. Licciardello, T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
[4] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).
[5] G. Parisi, Phys. Rev. Lett. **43**, 1754 (1979).
[6] D. S. Fisher and D. A. Huse, Phys. Rev. B **38**, 373 (1988).
[7] A. P. Ramirez, Ann. Rev. Mater. Sci. **24**, 453 (1994).
[8] R. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
[9] D. L. Shepelyansky, Phys. Rev. Lett. **70**, 1787 (1993).
[10] B. Derrida, Physics Reports **301**, 65 (1998); G. M. Schütz, in: *Phase Transitions and Critical Phenomena*, eds. C. Domb and J. L. Lebowitz, vol. 19 (Academic Press, San Diego, 2001).
[11] G. Tripathy and M. Barma, Phys. Rev. E **58**, 1911 (1998); M. Barma, Physica A **372**, 22 (2006).
[12] J. Krug, Braz. J. Phys. **30**, 97 (2000).
[13] C. Enaud and B. Derrida, Europhys. Lett. **66**, 83 (2004).
[14] T. E. Harris, J. Appl. Prob. **2**, 323 (1965); S. Alexander and P. Pincus, Phys. Rev. B. **18**, 2011 (1978); H. van Beijeren, K. W. Kehr, and R. Kutner, Phys. Rev. B **28**, 5711 (1983); R. Arratia, Ann. Probab. **11**, 362 (1983).
[15] D. G. Levitt, Phys. Rev. A **6**, 3050 (1973).
[16] Q. H. Wei, C. Bechinger, and P. Leiderer, Science **287**, 625 (2000).
[17] B. X. Cui, H. Diamant, and B. H. Lin, Phys. Rev. Lett. **89**, 188302 (2002).
[18] P. S. Burada, P. Hanggi, F. Marchesoni, G. Schmid, and P. Talkner, ChemPhysChem **10**, 45 (2009).
[19] Y. G. Sinai, Theor. Probab. Appl. **27**, 256 (1982).
[20] A. O. Golosov, Comm. Math. Phys. **92**, 491 (1984).
[21] H. Kesten, Physica A **138**, 299 (1986).
[22] E. Koscielny-Bunde, A. Bunde, S. Havlin, and H. E. Stanley, Phys. Rev. A **37**, 1821 (1988); H. E. Roman, M. Schwartz, R. B. Selinger, A. Bunde, and S. Havlin, Europhys. Lett. **7**, 389 (1988).
[23] D. S. Fisher, P. Le Doussal, and C. Monthus, Phys. Rev. Lett. **80**, 3539 (1998).
[24] J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990); G. Matheron and G. de Marsily, Water Resources Res. **16**, 901 (1980); S. Redner, Physica D **38**, 287 (1989).
[25] Taking logarithmic corrections into account, the second transition time is $t \sim \epsilon^{-4}[\ln(1/\epsilon)]^3$.
[26] The super-diffusive displacement (3) is smaller than the diffusive displacement, $\sigma \sim t^{1/2}$, up to $t \sim \epsilon^{-4}$.