Shock-Like Dynamics of Inelastic Gases

E. Ben-Naim†, S. Y. Chen†, G. D. Doolen†, and S. Redner‡
†*Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545*
‡*Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215*

We provide a simple physical picture which suggests that the asymptotic dynamics of inelastic gases in one dimension is independent of the degree of inelasticity. Statistical characteristics, including velocity fluctuations and the velocity distribution are identical to those of a perfectly inelastic sticky gas, which in turn is described by the inviscid Burgers equation. Asymptotic predictions of this continuum theory, including the $t^{-2/3}$ temperature decay and the development of discontinuities in the velocity profile, are verified numerically for inelastic gases.

PACS: 47.70.Nd, 45.70.Mg, 05.40.-a, 81.05.Rm

Gases of inelastically colliding particles model the dynamics of granular materials [1,2], geophysical flows [3], and large-scale structure of matter in the universe [4]. Typically, a fraction of the kinetic energy is dissipated in each collision, leading to interparticle velocity correlations, a clustering instability [5–9], and in the absence of external energy input, an inelastic collapse [10–15]. The last feature presents an obstacle to long-time simulations, as an infinite number of collisions occur in a finite time.

In this Letter, we propose that a freely evolving inelastic gas is asymptotically in the universality class of a completely inelastic, sticky gas. Specifically, the temperature decreases in time as t^{-2} over an intermediate range, but asymptotically decays as $t^{-2/3}$. To test this hypothesis, we employ a simulation in which collisions between particles with sufficiently small relative velocities are perfectly elastic. This method allows us to bypass the inelastic collapse and probe the asymptotic regime.

We consider N identical point particles undergoing inelastic collisions in a one-dimensional periodic system of length L. The particles have typical interparticle spacing $x_0 = L/N$ and their typical velocity is v_0. We employ dimensionless space and time variables, $x \to x/x_0$, and $t \to tv_0/x_0$, thereby rescaling the ring length to N. Inelastic and momentum conserving collisions are implemented by changing the sign of the relative velocity and reducing its magnitude by a factor $r = 1 - 2\epsilon$, with $0 \le r \le 1$, after each collision. It is convenient to view the particle identities as "exchanged" upon collision, so that in a perfectly elastic collision the particles merely pass through each other, while for a small inelasticity each particle suffers a small deflection. The outcome of a collision between a particle with velocity v and another particle with velocity u is therefore

$$v \to v - \epsilon(v - u). \tag{1}$$

The granular temperature, or velocity fluctuation, $T(t) = \langle v^2(t)\rangle - \langle v(t)\rangle^2$, can be estimated in the intermediate time regime by considering the outcome of a single collision under the assumption that the system remains homogeneous. In each such collision, the energy lost is $\Delta T \propto -\epsilon(\Delta v)^2$, with Δv the relative velocity, while the time between collisions is $\ell/\Delta v$. Assuming homogeneity, we neglect fluctuations in the mean-free path $\ell \cong 1$ and posit a single velocity scale so that $v \sim \Delta v \sim T^{1/2}$. The temperature therefore obeys the rate equation $dT/dt \propto -\epsilon T^{3/2}$ giving

$$T(t) \sim (1 + A\epsilon t)^{-2}, \tag{2}$$

with A a constant of order unity [1]. For small times $t \ll t_{\rm dissip} \sim \epsilon^{-1}$ dissipation is negligible and the temperature does not evolve – the gas is effectively elastic. For larger times, the dissipation leads to a $\epsilon^{-2}t^{-2}$ temperature decay.

However, this behavior *cannot* be valid asymptotically, as the temperature must decrease monotonically with increasing dissipation. Moreover, the temperature is bounded from below by that of the perfectly inelastic gas with a vanishing restitution coefficient, $r = 0$. For such a sticky gas, the temperature decays as $t^{-2/3}$ and the typical cluster mass grows as $t^{2/3}$ [16]. This behavior is reminiscent of diffusion-controlled two species annihilation, where a small reaction probability results in a homogeneous intermediate time regime in which the density follows a t^{-1} mean-field decay, even for low spatial dimension d. However, at long times single-species domains which are opaque to opposite-species particles form and a slower $t^{-d/4}$ density decay follows [17].

For the inelastic gas, we argue that the role of the reaction probability is played by ϵ. For small ϵ, a particle can penetrate through a domain of $N < N_c(\epsilon) \sim \epsilon^{-1}$ coherently-moving particles without experiencing a substantial deflection. The critical cluster size $N_c(\epsilon)$ may be estimated by considering a collision between a moving particle and a cluster of N stationary particles. From Eq. (1), each collision between the incident particle and the next particle in the cluster reduces the incident particle velocity by roughly ϵ. After N collisions the incident velocity is $v_N \approx 1 - N\epsilon$. For the particle to pass through the cluster, the number of particles must therefore be less than ϵ^{-1}. It is in this range of cluster sizes that the system remains spatially homogeneous and the mean-field decay $T \sim \epsilon^{-2}t^{-2}$ holds.

However, once the cluster size is larger than $N_c(\epsilon) \sim \epsilon^{-1}$, an incident particle is "absorbed" and the decay follows that of the perfectly inelastic gas. That is, domains larger than $N_c(\epsilon)$ are opaque and present an effective restitution coefficient $r_{\rm eff} \equiv 0$ to incident particles.

We argue that a similar sticking mechanism also governs cluster-cluster collisions. The crossover time $t_{\rm stick}$ between these two regimes is obtained by matching the intermediate and long-time temperature decays, $\epsilon^{-2}t^{-2}$ and $t^{-2/3}$, to give $t_{\rm stick}\sim\epsilon^{-3/2}$.

These arguments suggest the temperature decay

$$T(t)\sim\begin{cases}1 & t\ll\epsilon^{-1}\equiv t_{\rm dissip};\\ \epsilon^{-2}t^{-2} & \epsilon^{-1}\ll t\ll\epsilon^{-3/2}\equiv t_{\rm stick};\\ t^{-2/3} & \epsilon^{-3/2}\ll t\ll N^{3/2};\\ N^{-1} & N^{3/2}\ll t.\end{cases}\tag{3}$$

The last regime reflects the final state of a finite N-particle system, namely a single cluster of mass $m=N$, velocity $v\sim N^{-1/2}$, and therefore energy $T\sim v^2\sim N^{-1}$. This final velocity follows from momentum conservation in which the total momentum P is the sum of N individual random momenta of order unity. Consequently, $P\propto N^{1/2}$ and $v=P/m\sim N^{-1/2}$.

The above crossover picture applies equally well to moderately inelastic gases where both $t_{\rm dissip}$ and $t_{\rm stick}$ are of order unity and the asymptotic behavior is realized immediately. While weakly inelastic systems with a small number of particles $N<\epsilon^{-1}$ will avoid the clustering regime and follow the t^{-2} cooling law indefinitely, the $t^{-2/3}$ sticky gas regime is always reached in the thermodynamic limit, $N\to\infty$. Therefore, the $t^{-d/2}$ decay conjectured in [18,19] based on two- and three-dimensional simulations does not extend to lower dimensions.

To probe the long-time behavior, we performed numerical simulations of N particles which are initially equally spaced ($\Delta x=1$) and uniformly distributed (on $[-1,1]$) in velocity. We implemented an event-driven simulation, keeping the collision times always sorted to facilitate identification of the next event. To circumvent the inelastic collapse, elastic collisions were implemented whenever the relative velocity of the colliding particles fell below a pre-specified threshold, $\Delta v<\delta$ [3,20]. In fact, the restitution coefficient for deformable spheres does approach unity when $\Delta v\to 0$ as a consequence of the nonlinear Hertz contact law [21].

Fig. 1 shows that the temperature of the freely cooling inelastic gas asymptotically decays as $t^{-2/3}$, *independent* of the restitution coefficient. Moreover, the time scale over which this decay occurs diverges in the limit of vanishing dissipation. We also simulated the completely inelastic gas ($r=0$) where particles aggregate (and conserve momentum) upon collision. This gives the same asymptotic temperature decay as the partially inelastic gas. From Fig. 1, notice that the two crossover times $t_{\rm dissip}\sim\epsilon^{-1}$ and $t_{\rm stick}\sim\epsilon^{-3/2}$ are consistent with the data for the cases $r=0.9$ and 0.99, and that the temperature is of the appropriate order $T(t_{\rm stick})\sim\epsilon$ at the homogeneous-sticky crossover. On the other hand for $r=0.5$, the intermediate t^{-2} regime no longer exists and only the sticky gas behavior is realized.

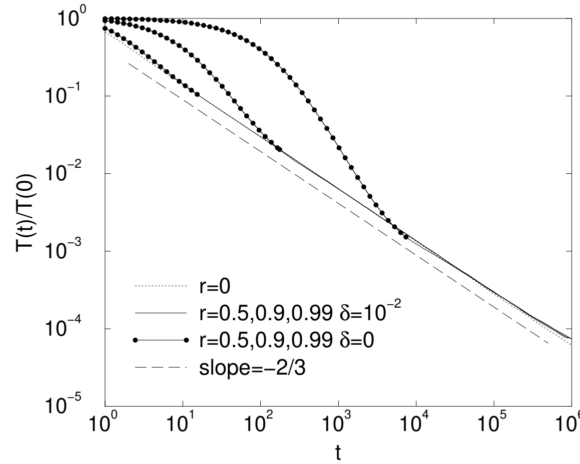

FIG. 1. Temperature $T(t)$ versus time for the freely cooling inelastic gas with restitution coefficients $r=0.5$, 0.9, and 0.99. The data represents averages over 10 realizations of $N=10^6$ particles and 10^4 collisions per particle. Also shown is a simulation for a sticky gas ($r=0$). A dashed line of slope $-2/3$ is plotted as reference. Least-square fits to the post-crossover data with velocity threshold $\delta=10^{-2}$ yield the decay exponents 0.67, 0.67, and 0.66 for $r=0$, 0.5, and 0.9 respectively.

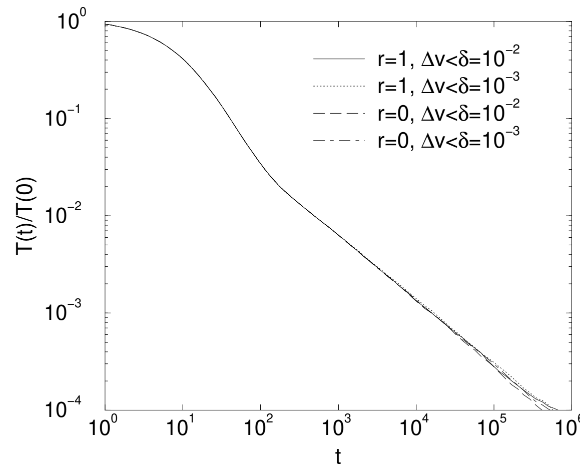

FIG. 2. Role of threshold velocity and the collision mechanism. Temperature decay for $r=0.9$, $N=10^6$ with different threshold velocities $\delta=10^{-2}$, $\delta=10^{-3}$, and different subthreshold collision mechanisms (both sticky and elastic).

The crossover between the homogeneous gas regime and the sticky gas regime is ultimately related to the inelastic collapse. In Fig. 1, simulation results for purely inelastic collisions are also shown (dots); these coincide with the results for the velocity threshold $\delta=10^{-2}$. However, as soon as the first cluster is formed, the number of collisions becomes infinite and the simulation does not proceed any further in time. Indeed, the last data point for these simulations marks the transition to the sticky gas regime. This provides an additional confirmation for the crossover picture, as the typical cluster mass of the sticky gas, $m(t_{\rm stick})\sim t_{\rm stick}^{2/3}\sim\epsilon^{-1}$, matches the critical mass for the appearance of opaque clusters.

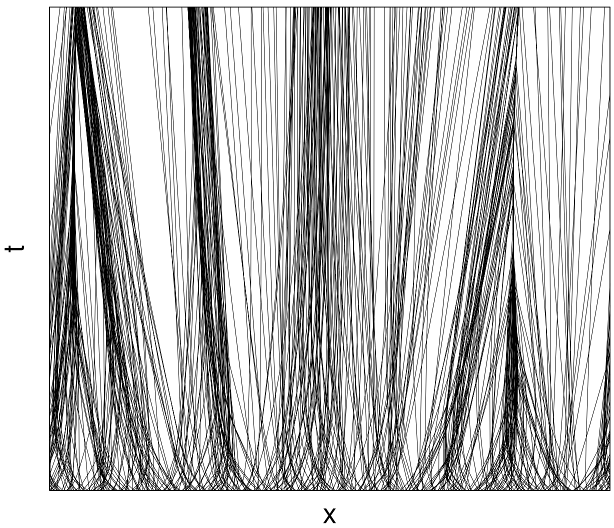

FIG. 3. Space-time evolution of a 500 particle system with $r = 0.9$ and $\delta = 10^{-2}$, up to $t = 600$.

To validate the simulation method, we checked that results are independent of the cutoff value (provided it is sufficiently small) as well as the sub-threshold collision mechanism (Fig. 2). In principle, the results can be trusted as long as the typical velocity is much larger than the cutoff, $v \sim t^{-1/3} \gg \delta$, *i.e.*, up to time $t_{\rm valid} \sim \delta^{-3}$. As shown in the figure, the results for $\delta = 10^{-2}$ and 10^{-3} nearly coincide until $t = 10^6$, consistent with our expectation. Furthermore, the space-time evolution of a weakly inelastic gas illustrates how aggregation eventually dominates (Fig. 3).

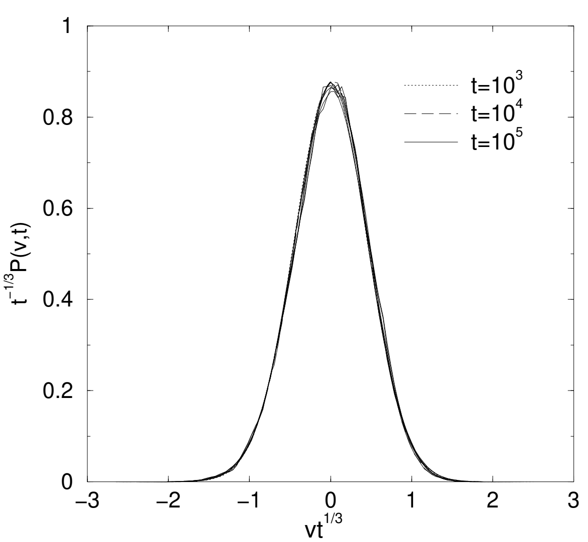

FIG. 4. Scaling of the velocity distribution for restitution coefficients $r = 0$, 0.5, and 0.9, at times $t = 10^3$, 10^4, 10^5. The nine data sets represent averages over 200 realizations in a system of $N = 10^5$ particles and with the cutoff $\delta = 10^{-3}$.

We now investigate whether the velocity distribution, and not merely the overall velocity scale, is also independent of r. We therefore computed this distribution for $r = 0.9$ (weakly inelastic), $r = 0.5$ (moderately inelastic), and $r = 0$ (perfectly inelastic) at three very different times which are well into the clustering regime. For the $r = 0$ case, the cluster velocity was weighted by the cluster mass, to compare with the $r > 0$ cases. As shown in Fig. 4, the normalized velocity distribution

$$P(v,t) \sim t^{1/3}\Phi(vt^{1/3}), \tag{4}$$

is described by an identical scaling function $\Phi(z)$ for these widely different values of r. This universality provides further confirmation that the asymptotic behavior for any $r < 1$ is governed by the $r = 0$ "fixed point".

Further insights about the behavior of the inelastic gas are provided by the connection to the Burgers equation [4,22]. Since sticky gases are described by the inviscid ($\nu \to 0$) limit of the Burgers equation

$$v_t + vv_x = \nu v_{xx}, \tag{5}$$

supplemented by the continuity equation $\rho_t + (\rho v)_x = 0$, we conclude that this continuum theory also describes the asymptotics of the inelastic gas in the thermodynamic limit. The Burgers equation may be reduced to the diffusion equation by the Hopf-Cole transformation $v = -2\nu(\ln u)_x$, and therefore is solvable. In our case, the relevant initial condition is delta-correlated velocities $\langle v_0(x)v_0(x')\rangle = \delta(x - x')$. The resulting velocity profile is discontinuous, and the corresponding shocks can be identified with clusters in the sticky gas. Indeed, both shock coalescence processes and cluster-cluster collisions in the sticky gas conserve mass and momentum.

The relation to the Burgers equation is useful in several ways. First, statistical properties of the shock coalescence process have been established analytically [23]. For example, the tail of the particle velocity distribution (4) is suppressed according to

$$\Phi(z) \sim \exp(-{\rm const.} \times |z|^3\,), \qquad |z| \gg 1. \tag{6}$$

This behavior can be understood by considering the density of the fastest (order unit velocity) particles. For such a particle to maintain its velocity to time t, it must avoid collisions. This requires that an interval of length $\propto t$ ahead of the particle must be initially empty [24]. For an initially random spatial distribution, the probability of finding such an interval decays exponentially with length; thus $P(1,t) \sim \exp(-{\rm const.} \times t)$. Using $\Phi(z) \sim \exp(-{\rm const.} \times |z|^\gamma)$ and $z = vt^{1/3}$ then yields $\gamma = 3$. Interestingly, over most of the range of scaled velocities, the numerically obtained velocity distribution deviates only slightly from a Gaussian, reflecting the small constant in (6) [23].

Another important prediction of Eq. (5) is that the velocity is linear in the Eulerian coordinate x and the Lagrangian coordinate $q(x,t)$

$$v(x,t) = \frac{x - q(x,t)}{t}. \tag{7}$$

This form also characterizes the asymptotic velocity profile of inelastic gases. Fig. 5 shows such a sawtooth velocity profile from an inelastic gas simulation. The slopes of the linear segments of the profile are consistent with the t^{-1} prediction of Eq. (7). The inelastic collapse is simply a finite time singularity characterized by the development of a discontinuity in the velocity profile, *i.e.*, a shock.

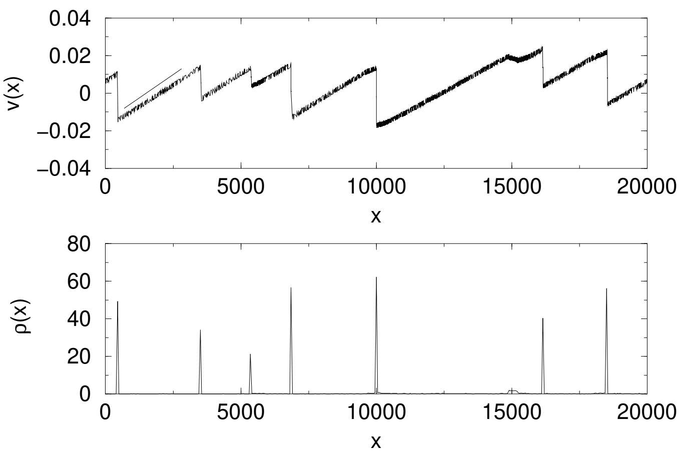

FIG. 5. Shock profile of an inelastic gas. Density and velocity are plotted at time $t = 10^5$ in a system with $N = 2\times10^4$ particles, $r = 0.99$ and $\delta = 10^{-4}$. A line with slope t^{-1} is plotted for reference. The number of shocks is consistent with the expected number $Nt^{-2/3} \cong 9$.

In higher dimensions as well, the temperature of an inelastic gas is a monotonically increasing function of r and hence, it is bounded from below by the $r = 0$ case. Therefore, we speculate that $r = 0$ remains the fixed point in higher dimensions. On the other hand, the Burgers equation $\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = \nu\nabla^2\mathbf{v}$ approximately describes the sticky gas in the limit $\nu \to 0$ [4]. The known $t^{-d/2}$ temperature decay of the Burgers equation [4], valid for $2 \le d \le 4$ (with possible logarithmic corrections at the crossover dimensions), then yields

$$T(t) \sim \begin{cases} 1 & t \ll \epsilon^{-1}; \\ \epsilon^{-2}t^{-2} & \epsilon^{-1} \ll t \ll \epsilon^{-4/(4-d)}; \\ t^{-d/2} & \epsilon^{-4/(4-d)} \ll t \ll N^{2/d}; \\ N^{-1} & N^{2/d} \ll t. \end{cases} \tag{8}$$

Interestingly, both the decay exponents [18,19], the formation of string-like clusters [5,11,20], and even the possibility of a percolating network of clusters [25], features that were found primarily numerically, are all predicted by the Burgers equation. Additionally, the critical cluster size increases with the dimension according to $N_c(\epsilon) \sim \epsilon^{-2d/(4-d)}$, suggesting that the inelastic collapse is avoided when $d > d_c = 4$, and that the homogeneous gas behavior $T \sim \epsilon^{-2}t^{-2}$ holds indefinitely above this critical dimension.

In summary, our results suggest that the asymptotic behavior of a one-dimensional inelastic gas with many particles is governed by the $r = 0$ sticky gas fixed point, and that the appropriate continuum theory is the inviscid Burgers equation. This connection provides several exact statistical properties of inelastic gases. Conversely, inelastic gases may provide a useful tool to study shock dynamics. The suggestive behavior of the inelastic gas in higher dimensions deserves careful investigation.

This research is supported by the Department of Energy under contract W-7405-ENG-36 (at LANL), and by the NSF under grant DMR9632059 (at BU). We thank G. P. Berman and P. L. Krapivsky for useful discussions.

[1] P. K. Haff, J. Fluid Mech. **134**, 401 (1983).
[2] L. P. Kadanoff, Rev. Mod. Phys. **71**, 435 (1999).
[3] C. S. Campbell, Annu. Rev. Fluid Mech. **22**, 57 (1990).
[4] S. F. Shandarin and Ya. B. Zeldovich, Rev. Mod. Phys. **61**, 185 (1989)
[5] I. Goldhirsch and G. Zanetti, Phys. Rev. Lett. **73** 1619 (1993).
[6] Y. Du, H. Li, and L. P. Kadanoff, Phys. Rev. Lett. **74**, 1268 (1995).
[7] E. L. Grossman, T. Zhou, E. Ben-Naim, Phys. Rev. E **55**, 4200 (1997).
[8] A. Kudrolli, M. Wolpert, J. P. Gollub, Phys. Rev. Lett. **78**, 1383 (1997).
[9] N. Sela and I. Goldhirsch, J. Fluid. Mech. **361**, 41 (1998).
[10] B. Bernu and R. Mazighi, J. Phys. A **23**, 5745 (1990).
[11] S. McNamara and W. R. Young, Phys Fluids A **4**, 496 (1992); **5**, 34 (1993). Phys. Rev. E **50**, R28 (1994).
[12] P. Deltour and J.-L. Barrat, J. Phys. I France **7**, 137 (1997).
[13] T. P. C. van Noije, M. H. Ernst, E. Trizac and I. Pagonabarraga, Phys. Rev. E **59**, 4326 (1999).
[14] T. P. C. van Noije and M. H. Ernst, cond-mat/9907012.
[15] J. J. Brey, F. Moreno and M. J. Ruiz-Montero, Phys. Fluids **10**, 2965 (1998); 2976 (1998).
[16] G. F. Carnevale, Y. Pomeau and W. R. Young, Phys. Rev. Lett. **64**, 2913 (1990).
[17] K. Kang and S. Redner, Phys. Rev. A **32**, 435 (1985).
[18] R. Brito and M. H. Ernst, Europhys. Lett. **43**, 497 (1998).
[19] S. Y. Chen, Y. Deng, X. Nie, and Y. Tu, cond-mat/9804235.
[20] S. Luding and S. McNamara, Gran. Matt. **1**, 3 (1998).
[21] R. Ramirez, T. Pöschel, N. V. Brilliantov, T. Schwager, cond-mat/9905382.
[22] J. M. Burgers, *The nonlinear diffusion equation*, Reidel, Dordrecht (1974).
[23] L. Frachebourg and P. A. Martin, cond-mat/9905056.
[24] E. Ben-Naim, P. L. Krapivsky, and S. Redner, Phys. Rev. E **50**, 822 (1994).
[25] J. Tobochnik, cond-mat/9905254.