Domain Statistics in Coarsening Systems

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We study the domain number and size distributions in the one-dimensional Ising and q-state Potts models subject to zero-temperature Glauber dynamics. The survival probability of a domain, $S(t) \sim t^{-\psi}$, and an unreacted domain, $Q_1(t) \sim t^{-\delta}$, are characterized by two independent nonrtrivial exponents. For the Ising case, we find $\psi = 0.126$ and $\delta = 1.27$ using numerical simulations. We develop an independent interval approximation (IIA) that predicts the qualitative behavior of the domain distribution and provides good estimates for the exponents. Exact results for the domain distribution are also obtained in several solvable cases.

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I. INTRODUCTION

When a system is quenched from a homogeneous high-temperature disordered state into a low-temperature state it does not order instantaneously; instead, domains of equilibrium ordered phases form on larger and larger scales [1]. It has been generally confirmed that a scale-invariant morphology is developed, i.e. the network of domains is (statistically) independent of time when lengths are rescaled by a single characteristic length scale that typically grows algebraically with time. However, even for simple coarsening processes little is known about more subtle properties such as the domain size distribution [2,3]. One such feature that has attracted considerable interest recently concerns the “persistence” of the local order parameter, the probability that it has not changed sign in a given time interval. Persistence has been investigated theoretically [4–9] and experimentally [10] in spin systems, interacting particles systems [11–14], Lotka-Volterra models [15,16], breath figures growth [17], foams [18], and even simple diffusion [19,20].

Single spin persistence provides a natural counterpart to the survival probability in the realm of many particle systems. In the context of reaction processes, persistence is equivalent to the survival of immobile impurities and therefore does not provide information about collective properties of the bulk. In contrast, domains are the natural spatial elements of a coarsening process. In this paper, we ask for example what is the survival of an entire domain, $S(t)$? This quantity decreases with time as a power-law $S(t) \sim t^{-\psi}$. Similar to other critical exponents, ψ is universal in the sense that it is independent of many details such as the initial conditions. However, it is model dependent and in this respect differs from the growth exponent that depends only on the conservative nature of the dynamics.

In the present work, we examine systems with short-range interactions described by a scalar non-conserved order parameter. We focus on the 1D Ising model and its generalization to the q-state Potts model [21] evolving according to Glauber spin-flip dynamics [22]. Sequential dynamics has been chosen without loss of generality as parallel dynamics exhibit similar asymptotic behavior [23]. In one-dimensional systems with short-range interactions, the order-disorder transition takes place at $T = 0$ [24], and since we are interested in coarsening, we restrict attention to zero temperature.

This paper is organized as follows. In the next section, we discuss the Ising and Potts models. We define the domain number distribution and determine it analytically in the limiting cases, $q \to \infty$ and $q \to 1$. Otherwise, we develop an Independent Interval Approximation (IIA) that assumes no correlations between adjacent domains. The IIA predictions compare well with Monte Carlo simulations by giving correct description of the domain statistics as well as good estimates for the underlying exponents. Section II is concluded with the $q = \infty$ Potts model in arbitrary dimension $d \geq 2$. In Sec. III, we obtain the domain distribution in two solvable cases: the Potts model with only energy lowering transitions and the deterministic ballistic annihilation model [25,15,16]. Section IV discusses some open issues and contains a summary.

II. ISING AND POTTS MODELS

A. The Ising model and the q-state Potts model

We start with the 1D Ising model subject to $T = 0$ Glauber dynamics [22]. To examine the role of the number of equilibrium phases we also consider a generalization of the Ising model, the q-state Potts model. Experimental realizations are known for $q = 2$ (the Ising model) and $q = 3, 4, \infty$ [21]. The $q = \infty$ case describes several cellular structures [26], e.g., polycrystals [27], foams [18], soap froth [28], and magnetic bubbles [29].

We consider uncorrelated initial conditions where each of the q phases is present with equal density $1/q$. The $T = 0$ Glauber-Potts dynamics proceeds by selecting a spin at random and changing its value to that of one of its randomly selected neighbors. Thus, domain walls perform a random walk and upon contact, they annihilate or coalesce, depending on the state of the corresponding domains [30–32]. Identifying a domain wall with a particle,

(A), and absence of a domain wall with a hole (0), one finds the single-species diffusion-reaction process [30–32]

$$\begin{aligned} A0 &\overset{\frac{1}{2}}{\rightleftharpoons} 0A, \\ AA &\xrightarrow{\frac{1}{q-1}} 00, \\ AA &\xrightarrow{\frac{q-2}{q-1}} A0 \text{ or } 0A. \end{aligned} \tag{1}$$

The rates indicate the relative probabilities by which each event occurs.

B. Domain Number Distribution: Definition and Scaling Properties

Our goal in this study is to investigate $S(t)$, the probability that a domain, initially present at the system at time $t = 0$, has not flipped up to time t (see Fig. 1). We will present theoretical and numerical evidence supporting an algebraic long time decay of this survival probability,

$$S(t) \sim t^{-\psi}. \tag{2}$$

Such a behavior is robust, as the exponent ψ is not sensitive to initial state of the system (provided long ranged correlations are absent). Our results will also strongly suggest that the exponent ψ is nontrivial, i.e., it cannot be extracted from so-far known exponents associated with the Ising model.

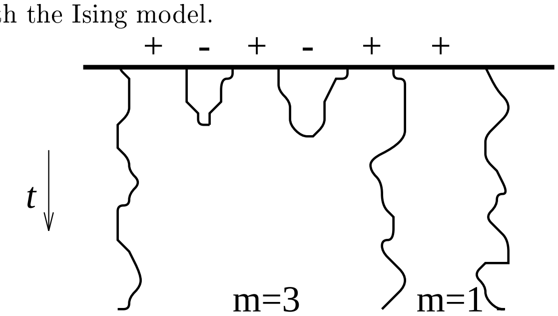

Fig. 1. Domain motion in the Ising model. Surviving domains are marked by +, annihilated domains by −. The domain number at a later time is also indicated.

In principle, a surviving domain may undergo coalescence with other similar phase domains. Thus, a natural generalization of the domain survival probability is $Q_m(t)$, the density of domains composed of m original domains (see Fig. 1). This quantity satisfies the initial condition $Q_m(0) = \delta_{m,1}$. The total domain density, $N(t)$, is given by $N(t) = \sum_m Q_m(t)$, while the domain survival probability counts *initial* domains that have not shrunk and hence contains the density $Q_m(t)$ with weight m

$$S(t) = \sum_m m Q_m(t). \tag{3}$$

The average number of domains contained within a surviving domain $\langle m(t)\rangle = S(t)/N(t)$ grows algebraically according to $\langle m(t)\rangle \sim t^{\nu-\psi}$ with ν the domain decay exponent, $N(t) \sim t^{-\nu}$. If the behavior of $Q_m(t)$ is truly self-similar, it should follow the scaling form

$$Q_m(t) \simeq t^{\psi-2\nu} \mathcal{Q}(m t^{\psi-\nu}). \tag{4}$$

The scaling function $\mathcal{Q}(z)$ exhibits the following extremal behavior

$$\mathcal{Q}(z) \sim \begin{cases} z^{\sigma} & z \ll 1, \\ \exp(-\kappa z) & z \gg 1. \end{cases} \tag{5}$$

The small argument tail describes domains that contain a very small number of initial domains. In particular, the quantity

$$Q_1(t) \sim t^{-\delta} \tag{6}$$

is of special interest: It gives the density of domains which avoided merging with their neighboring domains up to time t.

The inequalities $Q_1(t) \le \sum_m Q_m(t) \le \sum_m m Q_m(t)$ lead to the bounds

$$\psi \le \nu \le \delta. \tag{7}$$

Taking into account that at least one surviving domain surrounds a persistent spin gives $P(t) \le S(t)$, where $P(t) \sim t^{-\theta}$ is the density of persistent spins. Thus we arrive at another upper bound

$$\psi \le \theta \tag{8}$$

for the exponent ψ. We shall show below that these bounds are strict for the Potts model and that there are models with $\psi = \theta$, $\psi = \nu$, and $\nu = \delta$. The bounds (7) and (8) suggest that a domain decays with the slowest rate in the problem.

A useful relation between the scaling exponents can be obtained by substituting $m = 1$ in Eq. (4)

$$\delta - \nu = (\nu - \psi)(1 + \sigma). \tag{9}$$

Thus, among the three exponents ψ, δ, and σ, only two are independent. For the q-state Glauber-Potts model both the domain decay exponent $\nu = 1/2$ [22,30] and the persistence exponent $\theta(q)$ [6] are known. It gives hope that analytical determination of the domain exponents is also possible.

Quite obviously, domains disappear when their size vanishes, and therefore the domain size and number distributions are intimately related. One is therefore forced to consider the distribution of domains of size n consisting of m original domains at time t, denoted by $P_{n,m}(t)$. The aforementioned number distribution is $Q_m(t) = \sum_n P_{n,m}(t)$, and consequently, the domain survival probability is $S(t) = \sum_{n,m} m P_{n,m}(t)$.

As will be seen later, studying the joint size-number distribution requires detailed knowledge of the domain size distribution $P_n(t) = \sum_m P_{n,m}(t)$. This distribution obeys the normalization condition

$$\sum_n nP_n(t) = 1 \tag{10}$$

reflecting the conservation of the total length. The total domain density is simply $N(t) = \sum_n P_n(t)$. Since the average domain length grows as $n \sim t^\nu$, the length distribution follows the scaling form

$$P_n(t) \simeq t^{-2\nu}\mathcal{P}(nt^{-\nu}). \tag{11}$$

The length distribution scaling function has the following limiting behavior [2,3]

$$\mathcal{P}(x) \sim \begin{cases} x & x \ll 1, \\ \exp(-\lambda x) & x \gg 1. \end{cases} \tag{12}$$

In subsection D, we shall develop an approximation scheme that helps elucidate many of the qualitative and quantitative features of the domain size and number distributions.

C. Solvable Cases

1. The $q \to \infty$ Limit

In the $q = \infty$ case, similar phase domains never coalesce and therefore the domain number is trivial, $m = 1$. Thus $P_{n,m}(t) = P_n(t)\delta_{m,1}$, $N(t) = S(t) = Q_1(t)$, and $\nu = \psi = \delta = 1/2$. The value of the exponent has been obtained by noting that domain boundaries perform independent random walks and a domain disappears when its boundaries meet. Thus domains survive with probability identical to that of a random walk in the vicinity of a trap, $N(t) \sim t^{-1/2}$ [33], or $\nu = 1/2$. On the other hand, an individual up spin inside this domain has not changed its sign if it has not been crossed by both domain walls. Therefore, the persistence probability is proportional to $t^{-1/2} \times t^{-1/2} = t^{-1}$, i.e. $\theta = 1$. Hence, the bound (8) is strict.

The domain length distribution $P_n(t)$ obeys the diffusion equation

$$\frac{dP_n}{dt} = P_{n+1} + P_{n-1} - 2P_n, \tag{13}$$

with the boundary condition $P_0(t) = 0$. This rate equation satisfies the length conservation of Eq. (10). Solving (13) subject to the appropriate initial conditions, $P_n(0) = \delta_{n,1}$, gives

$$P_n(t) = [I_{n-1}(2t) - I_{n+1}(2t)]\exp(-2t), \tag{14}$$

and

$$N(t) = S(t) = Q_1(t) = [I_0(2t) + I_1(2t)]\exp(-2t), \tag{15}$$

where I_n is the modified Bessel function of order n [34]. The length distribution scales according to Eq. (11), with $\nu = 1/2$ and $\mathcal{P}(x) \simeq x\exp(-x^2/4)/\sqrt{\pi}$. The generic exponential behavior of Eq. (12) is now replaced by a Gaussian one, indicating that $\lambda \to 0$ as $q \to \infty$. In the long time limit, $S(t) = Q_1(t) = N(t) \simeq (\pi t)^{-1/2}$ confirming the previous heuristic findings $\nu = \psi = \delta = 1/2$.

2. The $q \to 1$ Limit

The 1D $T = 0$ Glauber-Potts model with arbitrary $q \geq 1$ can be mapped to the Ising model with magnetization $\mu = 2/q - 1$ [35]. In other words, the volume fraction of the down phase is $\varphi = 1 - 1/q$ [7]. In particular, the limit $\varphi \to 0$ allows treatment of the limiting case $q \to 1$ by focusing on the majority domains. The typical initial size of such domains is $\varphi^{-1} \to \infty$. This shows that in the limiting case $q = 1$ the minority domains cannot meet and the majority domains' sizes change appreciably due to coalescence. Thus, majority domains never disappear, i.e., $S(t) = 1$ and $\psi = 0$. Similarly, the persistence exponent is found: $\theta = 0$. A majority domain remains unreacted till time t if both of its neighboring minority domains survive, $Q_1(t) = N^2(t)$. The density is given by the $q = \infty$ solution (15), and we find $Q_1(t) \simeq (\pi t)^{-1}$ and $\delta = 1$.

The number distribution of the majority domains can be determined as well. The dynamics proceeds by minority domains shrinking to zero and thus leading to coalescence of surrounding majority domains. Such aggregation events occur independently with rate P_1/N^2 and the domain number distribution evolves according to

$$\frac{dQ_m}{dt} = \frac{P_1}{N^2}\left[\sum_{i=1}^{m} Q_j Q_{m-j} - 2NQ_m\right], \tag{16}$$

subject to the initial conditions $Q_m(0) = \delta_{m,1}$. It is helpful to absorb the time-dependent rate P_1/N^2 into the time variable

$$T = \int_0^t dt' \, \frac{P_1(t')}{N^2(t')} = N^{-1}(t) - 1, \tag{17}$$

with the overall density of (15), and the last equality evaluated using $\dot{N} = -P_1$. With this time variable Eq. (16) reduces to the classical Smoluchowski equation [36]

$$\frac{dQ_m}{dT} = \sum_{j=1}^{m} Q_j Q_{m-j} - 2NQ_m. \tag{18}$$

Solving Eq. (18) with the appropriate monodisperse initial conditions gives $\mathcal{Q}_m(T) = T^{m-1}(1+T)^{-m-1} \simeq T^{-2}\exp(-m/T)$ [36]. Indeed $Q_1(T) = N^2(T) = (1+T)^{-2}$, in agreement with the previous argument. In the long-time limit, $T \simeq N^{-1} \simeq \sqrt{\pi t}$ and $Q_m(t) \simeq$

$(\pi t)^{-1}\exp\left[-m(\pi t)^{-1/2}\right]$. Thus the domain number distribution scales according to Eq. (4) with the purely exponential scaling function

$$\mathcal{Q}(z) = \pi^{-1}\exp(-z\pi^{-1/2}). \tag{19}$$

The average domain properties are $S(t) = 1$, $N(t) \simeq (\pi t)^{-1/2}$, and $Q_1(t) = N^2(t) \sim (\pi t)^{-1}$ and the scaling exponents $\sigma = \psi = 0$, $\nu = 1/2$, $\delta = 1$.

Changes in the domain size due to domain wall diffusion are negligible here and the joint size-number distribution evolves according to

$$\frac{dP_{n,m}}{dT} = \sum_{i,j} P_{i,j}P_{n-i,m-j} - 2NP_{n,m}. \tag{20}$$

Eq. (20) generalizes the Smoluchowski equations for aggregation with two conservation laws [37]. Introducing the generating function $F(u,v,T) = \sum_{n,m} u^n v^m P_{n,m}(T)$ one can solve Eq. (20) for *arbitrary* initial conditions to find [37] $F(u,v,T) = F_0(u,v)(1+T)^{-1}[1+T-TF_0(u,v)]^{-1}$. In the present case, the appropriate initial conditions are $P_{n,m}(0) = \delta_{m,1}\varphi^2(1-\varphi)^{n-1}$ and hence $F_0(u,v) = uv\varphi^2[1-u(1-\varphi)]^{-1}$. Evaluating the limits $n\varphi \to n$, and $t \to \infty$, we arrive at the scaling form $P_{n,m}(t) \simeq t^{-5/4}\Phi(x,y)$, with the scaling variables $x = (m+n)(\pi t)^{-1/2}$, $y = (m-n)(\pi t)^{-1/4}$, and the scaling function

$$\Phi(x,y) = (\pi x)^{-1/2}\exp(-x - y^2/2x). \tag{21}$$

Instead of the naive scaling variables $nt^{-1/2}$ and $mt^{-1/2}$, unusual scaling variables underlie the scaling function (21). The former scaling variable x is just the sum of the naive scaling variables while the latter "diffusive" scale y is hidden. In this case, the domain length and the domain number are equivalent, and their underlying scaling functions are identical $\mathcal{P}(x) = \pi^{-1}\exp(-x\pi^{-1/2})$. Nevertheless, there is a considerable difference between the variables n and m as the latter is generally not a conserved quantity. Furthermore, as q decreases from ∞ to 1, the decay coefficient λ governing the domain length distribution increases from 0 to $\pi^{-1/2}$.

The above results apply for q close to unity as long as neighboring domains do not interact, i.e., as long as the diffusion time scale is smaller that the domain size, $\sqrt{t} \ll \varphi^{-1}$. Eventually, this no longer holds, and correlations between majority domains develop. Nevertheless, in the limit $q \to 1$ the equations (18) and (20) are *exact* since no correlations develop if none are present initially. Similar reasoning applies to several models where domains are immobile and merging occurs [4].

D. Independent Interval Approximation (IIA)

Ignoring correlations between neighboring domains allows us to develop an approximate theory for the time-evolution of the domain distribution. This so-called IIA proved useful in studies of related reaction-diffusion processes [2,19,30].

1. The Length Distribution

The joint number distribution requires knowledge of the length distribution and we start by deriving a master equation for $P_n(t)$. Consider first the Ising case. In an infinitesimal time interval Δt, the domain $P_n(t)$ changes according to

$$\begin{aligned} P_n(t+\Delta t) &= (1-2\Delta t)P_n(t) \\ &+ \Delta t\, P_{n-1}(t)\left[1 - \frac{P_1(t)}{N(t)}\right] + \Delta t\, P_{n+1}(t) \\ &- \Delta t\, P_n(t)\,\frac{P_1(t)}{N(t)} + \Delta t\, P_1(t)\sum_{i+j+1=n}\frac{P_i(t)}{N(t)}\frac{P_j(t)}{N(t)}, \end{aligned} \tag{22}$$

where $N(t) = \sum_n P_n(t)$ is the total domain density. The first term on the right-hand side of Eq. (22) counts for the probability that both domain walls do not hop. The next two terms describe gain due to diffusion, with the prefactor $(1 - P_1/N)$ in the second term to ensure that the hopping domain wall does not disappear. The forth term represents loss due to disappearance of the smallest domain, located on the boundary of our domain, while the final term accounts for gain due to domain merger.

Eq. (22) assumes that the sizes of adjacent domains are uncorrelated, and thus is mean-field in nature. In the limit $\Delta t \to 0$ the difference equations (22) turn into a system of differential equations:

$$\begin{aligned} \frac{dP_n}{dt} &= P_{n-1} + P_{n+1} - 2P_n \\ &+ \frac{P_1}{N^2}\left[\sum_{i=1}^{n-2} P_iP_{n-1-i} - N(P_n + P_{n-1})\right]. \end{aligned} \tag{23}$$

Eqs. (23) apply for $n = 1$ if we set $P_0 \equiv 0$. One crucial test is to verify the length conservation of Eq. (10). Another test is to sum all equations in (23) to get $\dot{N} = -2P_1$. This is an *exact* equation, since three domains disappear and one is born in each annihilation event.

Generally, in the Potts case, the domain size distribution evolves according to the rate equation

$$\begin{aligned} \frac{dP_n}{dt} &= P_{n-1} + P_{n+1} - 2P_n \\ &+ \frac{P_1}{(q-1)N^2}\left[\sum_{i=1}^{n-2} P_iP_{n-1-i} - N(P_n + P_{n-1})\right]. \end{aligned} \tag{24}$$

Indeed, collision of domain walls results in annihilation with probability $\frac{1}{q-1}$ or in coalescence with probability $\frac{q-2}{q-1}$. Only annihilation events affect the domain distribution and thus the $\frac{1}{q-1}$ prefactor of the annihilation term.

In the cases $q = 2$ and $q = \infty$, Eqs. (23),(13) are clearly recovered. In the limit $q \to 1$, only the reaction term survives, in agreement with Eq. (20). One can also verify that the total length is conserved and the total domain density decays according to the *exact* rate equation

$$\frac{dN}{dt} = -\frac{q}{q-1} P_1. \tag{25}$$

The diffusion term in Eq. (24) implies $\langle n(t)\rangle \sim t^{1/2}$, and since $\langle n\rangle \sim N^{-1}$ the correct decay exponent $\nu = 1/2$ [22,30] is recovered. In the following, we will need to determine the asymptotic prefactor A, $N(t) \sim At^{-1/2}$, $A = \int dx\, \mathcal{P}(x)$. The density rate equation (25) implies $P_1 \simeq \mathcal{P}'(0)t^{-3/2}$ with $\mathcal{P}'(0) = \frac{q-1}{2q}A$.

A quantitative analysis of Eq. (24) may be carried by treating the variable n as continuous. The quantity $\mathcal{P}(x)$ satisfies

$$\mathcal{P}'' + \frac{1}{2}(x\mathcal{P})' + \frac{q-2}{2q}\mathcal{P} + \frac{1}{2qA}\,\mathcal{P} * \mathcal{P} = 0, \tag{26}$$

where $\mathcal{P}' \equiv d\mathcal{P}/dx$ and $\mathcal{P} * \mathcal{P} \equiv \int_0^x dy\, \mathcal{P}(y)\mathcal{P}(x-y)$. The normalized Laplace transform of the scaling function $\mathcal{P}(x)$, $p(s) = A^{-1}\int_0^\infty dx\, e^{-sx}\, \mathcal{P}(x)$, obeys

$$\frac{dp}{ds} = \frac{p^2}{qs} + \left(2s + \frac{q-2}{qs}\right) p - \frac{q-1}{qs}, \tag{27}$$

subject to the boundary condition $p(0) = 1$. The transformation $p(s) = 1 - qs^2 - qs\frac{d}{ds}\ln y(s)$ reduces the Riccati equation (27) into the parabolic cylinder equation,

$$\frac{d^2y}{ds^2} + \left(1 + \frac{2}{q} - s^2\right) y = 0. \tag{28}$$

The solution to (28) reads $y(s) = C_- D_{1/q}(-s\sqrt{2}) + C_+ D_{1/q}(s\sqrt{2})$, with $D_{1/q}(x)$ the parabolic cylinder function of order $1/q$ [34]. The large s behavior of $p(s)$, $p(s) \sim \frac{q-1}{2q}s^{-2}$, implies $C_- = 0$, and we get

$$p(s) = 1 - qs^2 - qs\frac{d}{ds}\ln D_{1/q}(s\sqrt{2}). \tag{29}$$

The normalization condition $\sum_n nP_n(t) = 1$ can be reduced to $Ap'(0) = -1$. This allows us to determine the constant

$$A = \frac{\Gamma\left[1 - \frac{1}{2q}\right]}{\Gamma\left[\frac{1}{2} - \frac{1}{2q}\right]}, \tag{30}$$

where Γ denotes the gamma function. In deriving (30) we have used the properties [34]

$$D_c(x) \sim x^c \exp(-x^2/4)[1 + \mathcal{O}(x^{-2})], \tag{31}$$

and

$$D_c(0) = \frac{\pi^{1/2}2^{c/2}}{\Gamma(1/2 - c/2)}, \quad D_c'(0) = -\frac{\pi^{1/2}2^{(c+1)/2}}{\Gamma(-c/2)}. \tag{32}$$

The value of the prefactor A predicted by the IIA may be compared to the exact one, $A_{\rm exact} = (1 - q^{-1})/\sqrt{\pi}$ [32] (see Fig.2). In the extreme cases of $q = 1$ and $q = \infty$ the prefactor A is exact. The mismatch is worst for the Ising ($q = 2$) case where $A = \Gamma(3/4)/\Gamma(1/4) \cong 0.337989$ while $A_{\rm exact} = (4\pi)^{-1/2} \cong 0.28209$ [22].

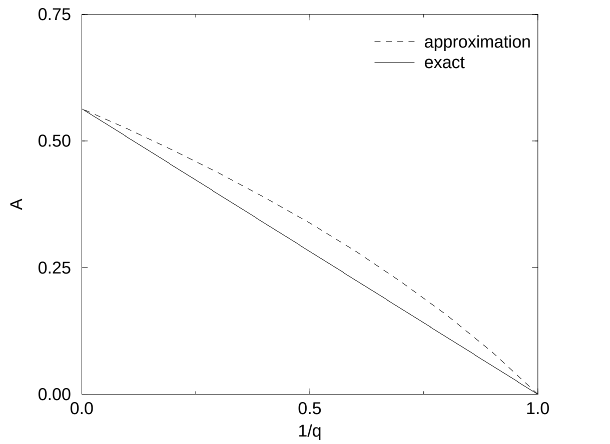

Fig. 2. The prefactor A of Eq. (30) vs. the exact value $A_{\rm exact} = (1 - q^{-1})/\sqrt{\pi}$.

The IIA gives correct qualitative results including: (i) density decay $N(t) \sim t^{-1/2}$; (ii) linear small size distribution, $\mathcal{P}(x) \simeq \frac{A(q-1)}{2q}x$, as can be seen by considering the large s behavior of $p(s) \simeq \frac{(q-1)}{2q}s^{-2}$; (iii) exponential tail, $\mathcal{P}(x) \simeq qA\lambda\exp(-\lambda x)$ as in Eq. (12). The tail follows from the behavior of the Laplace transform $p(s) \simeq q\lambda/(s+\lambda)$ near its pole at negative $s = -\lambda$, given by the first zero of $D_{1/q}(-\lambda\sqrt{2}) = 0$. For the Ising case one has $\lambda = 0.5409$. This value should be compared with the exact value $\lambda = \zeta(3/2)/4\sqrt{\pi} = 0.368468$ obtained by Derrida and Zeitak [3] and the approximate value $\lambda = 0.35783$ obtained by Alemany and ben-Avraham [2].

One may wonder regarding the value of the IIA. It is analogous to that of Ref. [2] but the quantitative agreement is worse in our case. The answer is simple: Our approach is *self-consistent* while the approach of Ref. [2] is not. Indeed, making use of the assumption that the domain sizes are uncorrelated, Alemany and ben-Avraham express the equal-time two-spin correlation function via the domain size distribution $P_n(t)$. Then they use the exact expression for the equal-time two-spin correlation function [22] to determine $P_n(t)$. However, would they use their key assumption that the domain sizes are uncorrelated everywhere they would eventually obtain our expression for $P_n(t)$. In contrast, our approach is self-consistent as all our the results are derived within the same scheme. Additionally, to determine the more subtle characteristics to be described below one does not enjoy the luxury of known exact analytical results.

2. The Joint Distribution

We are now in a position to tackle the joint size-number distribution, $P_{n,m}(t)$, which captures both the spatial and "historical" characteristics of the coarsening domain mosaic. The corresponding rate equation is a generalization of Eq. (24)

$$\frac{dP_{n,m}}{dt} = P_{n-1,m} + P_{n+1,m} - 2P_{n,m} \tag{33}$$
$$+\frac{P_1}{(q-1)N^2}\left[\sum_{i,j} P_{i,j}P_{n-1-i,m-j} - N(P_{n,m}+P_{n-1,m})\right]$$

with the initial condition $P_{n,m}(0) = \delta_{n,1}\delta_{m,1}$ and the boundary condition $P_{0,m}(t) = 0$. The variable m is "mute" in some sense. It appears in a nontrivial way only in the convolution term. One should verify that this master equation is self-consistent. First, by summing over m, we recover Eq. (24). Second, it implies that the domain survival probability satisfies the exact linear equation $dS/dt = -\sum_m mP_{1,m}$. So far, we have not succeeded in solving the joint distribution. Nevertheless, it is still possible to obtain analytically many interesting properties of Eq. (33), including the scaling exponents.

Let us consider the distribution of domains which have not merged with other domains up to t, $R_n(t) \equiv P_{n,1}(t)$. For such domains, the convolution term vanishes and they evolve according to the linear rate equation

$$\frac{dR_n}{dt} = R_{n-1} + R_{n+1} - 2R_n - \frac{P_1}{(q-1)N}(R_n + R_{n-1}) \tag{34}$$

with the initial condition $R_n(0) = \delta_{n,1}$ and the boundary condition $R_0(t) = 0$. In the continuum limit we again replace $R_{n-1}+R_{n+1}-2R_n$ by $\partial^2 R/\partial n^2$ and $R_n + R_{n-1}$ by $2R_n$ to find a diffusion-convection equation for $R_n(t)$. The transformation $R_n \to \tilde{R}_n N^{-2/q}$ reduces this equation to the diffusion equation (13) for $\tilde{R}_n$, which is solved to yield $R_n(t) \simeq N^{2/q}t^{-1}\mathcal{R}(nt^{-1/2})$, with $\mathcal{R}(x) = x\exp(-x^2/4)/\sqrt{\pi}$. The large n behavior of $P_{n,1}(t)$ mimics the $q=\infty$ case in that it exhibits a Gaussian behavior, $P_{n,1}(t) \sim \exp(-n^2/4t)$, while the average domain density decays exponentially, $P_n(t) \sim \exp(-n/\sqrt{t})$. Large intervals are more strongly suppressed when they have a small number of ancestors and therefor, the n-tail of the joint distribution strongly depends upon m. The total density of unreacted domains is $Q_1(t) = \sum_n R_n \sim t^{-\frac{1}{2}-\frac{1}{q}}$, which gives the decay exponent

$$\delta = \frac{1}{2} + \frac{1}{q}. \tag{35}$$

Obtaining the second independent exponent ψ is more involved. The natural approach, i.e., a direct investigation of the domain number distribution, Q_m, appears to be useless, as it requires knowledge of $P_{1,m}$ and hence the entire $P_{n,m}$. The domain survival probability can be alternatively obtained by considering $U_n(t) = \sum_m mP_{n,m}(t)$. This quantity obeys

$$\frac{dU_n}{dt} = U_{n-1} + U_{n+1} - 2U_n \tag{36}$$
$$+\frac{P_1}{(q-1)N^2}\left[2\sum_{i=1}^{n-2} U_i P_{n-1-i} - N(U_n + U_{n-1})\right],$$

obtained by summing Eqs. (33). We write $U_n(t)$ in a scaling form $U_n(t) \simeq t^{-\psi-1/2}\mathcal{U}(nt^{-1/2})$. Asymptotically, the domain survival probability reads $S(t) \simeq Bt^{-\psi}$ with $B = \int dx\, \mathcal{U}(x)$. The scaling distribution satisfies

$$\mathcal{U}'' + \frac{1}{2}(x\mathcal{U})' + \left(\psi - \frac{1}{q}\right)\mathcal{U} + \frac{1}{qA}\mathcal{U} * \mathcal{P} = 0. \tag{37}$$

The normalized Laplace transform of the scaling function $\mathcal{U}(x)$, $u(s) = B^{-1}\int_0^\infty dx\, e^{-sx}\mathcal{U}(x)$, obeys

$$\frac{du}{ds} = 2\left(\frac{p(s)+q\psi-1}{qs} + s\right)u - \frac{2\psi}{s}, \quad u(0) = 1. \tag{38}$$

In deriving (38) we used the relation $\mathcal{U}'(0) = B\psi$, found by integration of Eq. (37), combined with $A = \int dx \mathcal{P}(x)$. Substituting the explicit expression (29) for $p(s)$ into Eq. (38), and solving for $u(s)$ yields

$$u(s) = 2\psi s^{2\psi} D_{1/q}^{-2}(s\sqrt{2}) \int_s^\infty dr\, r^{-2\psi-1} D_{1/q}^2(r\sqrt{2}). \tag{39}$$

This solution is consistent with the anticipated $s\to\infty$ behavior, $u(s) \simeq \psi s^{-2}$. Furthermore, evaluating Eq. (39) near the origin gives $u(s) = 1 + F(\psi)s^{2\psi} + Cs + \cdots$. Therefore for $u'(s)$ to be finite near $s=0$, we must have $F(\psi) = 0$. Evaluating $F(\psi)$ gives

$$0 = \int_0^\infty dr\, r^{-2\psi} D_{1/q}(r) D'_{1/q}(r), \tag{40}$$

Interestingly the second domain survival exponent, ψ, is irrational, in contrast with δ.

For completeness, we write the leading extremal behavior of the function $U(x)$:

$$U(x) \sim \begin{cases} x & x \to 0; \\ x\exp(-\lambda x) & x\to\infty. \end{cases} \tag{41}$$

This behavior can be easily obtained from the extremal behavior of the function $u(s)$. When $s\to\infty$, $u(s) \sim s^{-2}$, while near the pole $s \to -\lambda$, one finds $u(s) \sim (s+\lambda)^{-2}$.

E. Numerical Results

To test the IIA predictions, we performed numerical simulations on a spin chain of size $L = 10^7$. Random initial conditions and periodic boundary conditions were used. The simulation data represents an average over 10 different realizations. For the Ising case, we found the exponent values $\psi = 0.126(1)$ and $\delta = 1.27(2)$ (see Figs. 3 and 4). These values should be compared with the IIA predictions of $\psi = 0.136612$ and $\delta = 1$.

As was the case for the persistence exponent, θ, the domain exponents strongly depend on q. Numerical values of the exponents ψ and δ are summarized in Table 1 for representative values of q. As q increases, the approximation improves and eventually becomes exact for the extreme case $q = \infty$. Thus, ψ is overestimated by up to 10% and δ is underestimated by up to 25%. Hence, domains of average number m are better approximated in comparison with domains with extremely small m. Although the estimates are not exact they are still useful as they exhibit the correct q-dependence. In the $q \to \infty$ limit, the exponents approach the limiting value 1/2 according to $\delta = \frac{1}{2} + \frac{1}{q}$ derived in Eq. (35), and $\psi \cong \frac{1}{2} - \frac{1}{q}$. The leading behavior for ψ follows directly from the scaling relation (9), δ and $\sigma(\infty) = 0$.

In the $q \to 1$ limit analysis of Eq. (40) suggests that ψ vanishes according to $\psi \propto (q-1)^2$. This behavior agrees with the $q = 1$ exact solution and is consistent with simulation results of an Ising chain with magnetization $\mu = 2/q - 1$. It is practically impossible to obtain δ conclusively because unreacted domains decay quickly. The limiting value as $q \to 1$ appears to be larger than the value suggested by the IIA, $\delta \cong 3/2 - (q-1)$.

We performed several checks to verify that the asymptotic behavior of Eq. (2) and Eq. (6) is robust. For example, it is independent of the initial domain wall concentration (provided that the correlations in the initial condition are short range). We conclude that ψ and δ are nontrivial exponent, i.e., they cannot be extracted from the known exponents associated with the Ising-Glauber model. Similar to the persistence exponent, $\theta(q)$, the exponents appear to be irrational except for the limiting cases $q = \infty$ ($\psi = \delta = \frac{1}{2}$, $\sigma = 0$) and, maybe, for $q = 2$ ($\psi = \frac{1}{8}$, $\delta = \frac{5}{4}$, $\sigma = 1$).

The numerical simulations also confirm that the distribution function $Q_m(t)$ scales according to Eq. (4) (see Fig. 5). The scaling function $\mathcal{Q}(z)$, defined in Eq. (5), decays exponentially for large argument and is algebraic for small argument. The scaling relations combined with the simulation values give $\sigma = 1.05(5)$. This is consistent with the linear behavior seen in Fig. 5 for $z \ll 1$. Hence, similar scaling functions underlie the domain number and size distributions [2,3]. As q increases from 2 to ∞, the exponent σ decreases from 1 to 0, respectively.

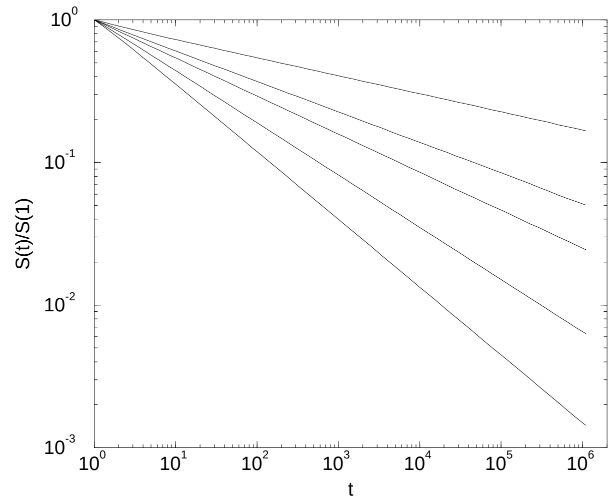

Fig. 3. The domain survival probability $S(t)/S(t=1)$ in the Potts model. Shown are Monte Carlo simulation results for $q = 2, 3, 4, 8$ and 50.

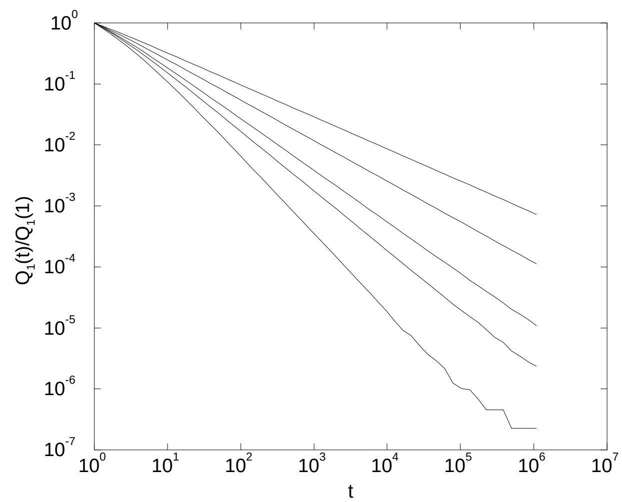

Fig. 4. The density of unreacted domains $Q_1(t)/Q_1(t=1)$ in the Potts model. The same Monte Carlo simulation as in Fig. 3.

	MC			Eq. (33)	
q	ψ	δ	σ	ψ	δ
2	0.126	1.27	1.05	0.136612	1
3	0.213	0.98	0.67	0.231139	5/6
4	0.267	0.85	0.50	0.287602	3/4
8	0.367	0.665	0.24	0.385019	5/8
50	0.476	0.525	0.03	0.480274	13/25
∞	1/2	1/2	0	1/2	1/2

Table 1: Domain exponents for the q-state Potts model in one dimension. Local slopes analysis was applied to the simulation data. The theoretical ψ is from Eq. (40) and $\delta = \frac{1}{2} + \frac{1}{q}$.

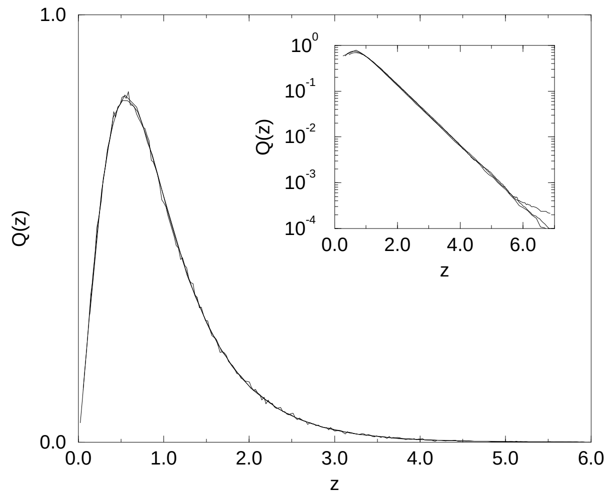

Fig. 5. The scaling distribution $\mathcal{Q}(z)$ vs. z for three different times $t = 10^2, 10^3, 10^4$. The data represents an average over 100 systems of size $L = 10^5$. The inset demonstrates the exponential behavior of the large-z tail.

Direct numerical integration of Eq. (33) reveals a number distribution, $Q_m(t)$, that scales according to Eq. (4), and has an exponential tail in agreement with the simulation results. Moreover, the emerging $S(t)$ falls within 5% of the actual survival probability over a significant temporal range, $t < 10^3$. In summary, in addition to predicting the correct scaling behavior, Eq. (33) provides a good approximation for many quantitative features of the domain distribution, and in particular, good estimates for the decay exponents.

F. $q = \infty$ Potts Model in Higher Dimensions

So far, our discussion has been restricted to one dimension. Domains are not necessarily well-defined in higher dimensions. In particular, in the Ising case it is not clear whether our results can be properly generalized to higher dimensions. However, in the $q \to \infty$ limit of the Potts model domains are well defined and dynamics considerably simplifies [35]. In this limit, it has been argued heuristically and confirmed numerically [38,18] and experimentally [39] for evolving soap froth, that $\theta = 1$ in two dimensions.

We now present a simple heuristic argument which gives the exponents $\psi = \delta = \theta = d/2$ for the $q = \infty$ Potts model in $d \geq 2$ dimensions. First we note that an exact correspondence between the dynamics of the q-state Potts model and the Ising model with magnetization $\mu = 2/q - 1$ holds only in one dimension. This happens due to the *global* conservation of the magnetization [22], a peculiar property of the 1D Ising model with zero-temperature Glauber dynamics. This global conservation for the locally non-conserved dynamics does not happen when $d > 1$, as seen by considering a single up spin in the sea of down spins. On the other hand, for the Potts model with symmetric initial conditions the density of any phase is globally conserved. This suggests a correspondence between the Potts model and the Ising model with globally conserved dynamics. It appears difficult to make such correspondence rigorous, although it is supported by several tests [35]. The reduction to the Ising model with magnetization $\mu = 2/q - 1$ in d dimensions can be hardly considered as a simplification except for the $q = \infty$ case which may be analyzed within the framework of the Lifshitz-Slyozov theory [1]. Indeed, as $q \to \infty$ the minority (up) phase approaches infinitesimal concentration, so up domains do not interact and the Lifshitz-Slyozov approach, suitably modified for the present case [35], should be exact.

So consider a set of bubbles (the domains tend to become round in high dimensions) of up phase in a sea of down phase. We cannot restrict ourselves to the single-domain situation as in one dimension since when $d > 1$ a single bubble would not evolve, while the set of bubbles do evolve: small bubbles shrink and large bubbles grow. We are not interested in details of the bubble evolution; the only relevant feature is that the radii distribution scales asymptotically, $N(R,t)$ $N(R,t) \simeq \mathcal{R}^{-(d+1)}\mathcal{N}(R/\mathcal{R})$, with the average radius $\mathcal{R} \sim \sqrt{t}$. This behavior is due to the nature of non-conserved dynamics [1] and the prefactor guarantees a conserved magnetization. Clearly, $S(t)$ is determined by computing the number of surviving bubbles, $S(t) \sim \int dR N(R,t) \sim \mathcal{R}^{-d}$, implying

$$\psi = \delta = \theta = d/2. \tag{42}$$

III. EXACTLY SOLVABLE CASES

Given that obtaining the exact domain distribution in the q-state Potts model appears to be a difficult problem, it might prove useful to study simpler problems which are exactly solvable. We present in this section exact results for a variant of the Potts model with simplified dynamics and for ballistic annihilation.

A. Diffusionless Dynamics

Consider the $T = 0$ q-state Potts with simplified dynamics where only energy lowering transitions are allowed. Thus, domain wall diffusion $A0 \rightleftharpoons 0A$ in Eq. (1) is forbidden and the reaction scheme is

$$\begin{aligned} AA &\xrightarrow{\frac{1}{q-1}} 00, \\ AA &\xrightarrow{\frac{q-2}{q-1}} A0 \text{ or } 0A. \end{aligned} \tag{43}$$

When $q = 2$, exchanging the roles of domain walls and vacant sites, this problem is equivalent to random sequential adsorption of dimers. Similarly, the $q = \infty$ case reduces to monomer adsorption subject to a volume exclusion constrain [40].

Assuming that neighboring intervals are uncorrelated, the domain length density rate equation reads

$$\frac{dP_n}{dt} = \frac{P_1}{(q-1)N^2}\left[\sum_{i=1}^{n-2} P_i P_{n-1-i} - 2NP_n\right] + \frac{(q-2)P_1}{(q-1)N}\left[P_{n-1} - P_n\right] - \delta_{n,1}P_1. \qquad (44)$$

For simplicity, we consider the antiferromagnetic initial condition $P_n(0) = \delta_{n,1}$. While the annihilation term is similar to Eq. (24), coalescence events are no longer offset by domain wall diffusion and thus the second term which is proportional to the coalescence probability $(q-2)/(q-1)$. It can verified that the total length $\sum_n nP_n = 1$ is conserved.

The density decays according to the familiar rate equation (25) $\dot{N} = -\frac{q}{q-1}P_1$. On the other hand, the minimal gap density satisfies $\dot{P}_1 = -P_1\left[1 + \frac{qP_1}{(q-1)N}\right]$. It is useful to introduce the normalized quantity $p_1 = P_1/N$ which obeys $\dot{p}_1 = -p_1$ and thus $p_1 = e^{-t}$. Using this result, the minimal gap density and hence the density is found

$$N(t) = \exp[-q(1-e^{-t})/(q-1)]. \qquad (45)$$

This agrees with known exact results for the $q = 2$ and $q = \infty$ cases [41–43]. Usually in random sequential adsorption problems, it is convenient to study the complementary density of gaps between domains which satisfies a *linear* rate equation. Nevertheless, the IIA is exact in this case as no "mixing" of domains due to diffusion occurs. The final domain density is given by $N(\infty) = \exp[-q/(q-1)]$. The systems quickly reaches a jamming configuration where domain walls are isolated and immobile. Thus, no coarsening occurs and the postulated scaling behavior does not apply. Nevertheless, as will be shown below, the domain size and number distributions and in particular their tails do resemble their diffusive counterparts.

The length distribution P_n can be found using normalized distribution $p_n = P_n/N$ which satisfies (for $n \geq 2$)

$$\frac{dp_n}{dt} = \frac{p_1}{q-1}\left[\sum_{i=1}^{n-2} p_i p_{n-1-i} + (q-2)p_{n-1}\right]. \qquad (46)$$

This equation can be further simplified by introducing the modified time variable T, defined via $dT/dt = p_1$ implying $T = 1 - e^{-t}$. To solve Eq. (46) we introduce the generating functions

$$p(z,T) = \sum_{n=1}^{\infty} p_n(T) z^n, \qquad (47)$$

that satisfies

$$\frac{dp(z,T)}{dT} = \frac{z}{q-1}\left[p(z,T)^2 + (q-2)p(z,T)\right] - z. \qquad (48)$$

Solving Eq. (48) subject to the monodisperse initial conditions $P_n(0) = \delta_{n,1}$, i.e., $p(z, T=0) = z$, we get

$$p(z,T) = 1 + q\left[\frac{z+q-1}{z-1}\exp\left(-\frac{qzT}{q-1}\right) - 1\right]^{-1}. \qquad (49)$$

Clearly, quantities such as the domain density and the domain length distribution approach exponentially fast their limiting values. We are especially interested in these limiting values. By expanding the generating functions in powers of z we get the limiting density of short domains $p_n(\infty) = 0$, $\frac{q-2}{2(q-1)}$, $\frac{q^2-3q+3}{3(q-1)^2}$, and $\frac{q^2(q-2)}{8(q-3)^3}$ for $n = 1,2,3,4$.

Similar to the behavior seen in the Glauber-Potts model, large domains are suppressed exponentially,

$$P_n(t) \sim [\lambda(q,t)]^n \qquad n \gg 1. \qquad (50)$$

Here $\lambda(q,t)$ is equal to the inverse of the first simple pole of the generating functions $p(z,t)$. This can be seen from the $\sum_n \lambda^n z^n \propto (\lambda^{-1} - z)^{-1}$. Note that $\lambda(q,t)$ vanishes when $t \to 0$ as implied by the initial conditions. When $t \to \infty$, $\lambda(q,t) \to \lambda_\infty(q)$. In particular when $q \to \infty$, $\lambda_\infty(q)$ vanishes according to $\lambda_\infty(q) \sim 1/(\ln q)$, indicating a faster than exponential limiting behavior. Indeed, when $q = \infty$ the generating functions $p(z,\infty) = 1 + (z-1)e^z$ gives an inverse factorial decay, $p_n(\infty) = (n-1)/n! \sim e^{-n\ln n}$. In the complementary $q = 1$ limit, the density vanishes, as was the case in the diffusive counterpart.

The domain number-size distribution can be obtained by generalizing Eq. (44)

$$\frac{dP_{n,m}}{dt} = \frac{P_1}{(q-1)N^2}\left[\sum_{i,j} P_{i,j}P_{n-1-i,m-j} - 2NP_{n,m}\right] + \frac{(q-2)P_1}{(q-1)N}\left[P_{n-1,m} - P_{n,m}\right] - \delta_{n,1}\delta_{m,1}P_1. \qquad (51)$$

A solution using the generating functions technique is possible here as well. However, this solution is too cumbersome, and we briefly discuss its qualitative features. There are two limiting cases. When $q = \infty$, the joint domain-number distribution simplifies to $P_{n,m} = P_n\delta_{m,1}$. When $q = 2$, domains are always of odd length and $P_{n,m} = P_n\delta_{m,(n+1)/2}$. Hence, the domain number distribution also decays exponentially, $Q_m(t) \sim [\Lambda(q,t)]^n$. Similar to the length distribution, the decay constant $\Lambda(q)$ vanishes when $q \to \infty$.

In summary, although the restricted dynamics Potts model does not exhibit coarsening or scale invariance, the number and size distributions mimic their diffusive counterpart large n and large q behavior.

B. Ballistic Annihilation Model

Consider a binary reaction process with particles moving ballistically and annihilating upon collision. Assuming a bimodal velocity distribution, we set these velocities equal to ± 1, without loss of generality. Identifying domain walls with particles, this two-velocity ballistic annihilation process [25] is equivalent to deterministic coarsening in a system with three equilibrium states [15,16].

The domain size distribution for this ballistic annihilation process has been investigated in [16]. Here we want to compute the domain survival probability $S(t)$. There are actually four such survival probabilities depending on the initial velocities of boundary interfaces; we denote the corresponding survival probabilities by $S_{++}(t), S_{+-}(t), S_{-+}(t)$, and $S_{--}(t)$. Then the total survival probability is just the sum $S(t) = \frac{1}{4}\left[S_{++}(t) + S_{+-}(t) + S_{-+}(t) + S_{--}(t)\right]$. We need to specify the initial conditions. Let us assume that interfaces are located according to the Poisson distribution with unit density. For such symmetric initial conditions we have $S_{++}(t) = S_{--}(t)$. One immediately gets the survival probability in the simplest case when the interfaces move toward each other:

$$S_{+-}(t) = e^{-2t}. \tag{52}$$

To compute the survival probability of parallel moving interfaces we note that the probability $S_+(t)$ for a single right-moving interface to survive is $S_+(t) = [S_{++}(t) + S_{+-}(t)]/2$. Combining the known result [25]

$$S_+(t) = S_-(t) = e^{-2t}[I_0(2t) + I_1(2t)], \tag{53}$$

with Eq. (52) we arrive at

$$S_{++}(t) = S_{--}(t) = e^{-2t}[2I_0(2t) + 2I_1(2t) - 1]. \tag{54}$$

Using the asymptotic behavior [34] $I_n(t) \simeq e^t/\sqrt{2\pi t}$ when $t \to \infty$, we get $S_{++}(t) \simeq 2/\sqrt{\pi t}$.

We turn now to the more challenging problem, i.e. to computation of the survival probability when the interfaces move away from each other. The final answer is relatively simple:

$$S_{-+}(t) = e^{-2t}[2I_1(2t) + 4I_2(2t) + 2I_3(2t) + 1] \tag{55}$$

so that the total survival probability reads

$$S(t) = \frac{1}{2}\, e^{-2t}[2I_0(2t) + 3I_1(2t) + 2I_2(2t) + I_3(2t)]. \tag{56}$$

Eqs. (55) and (56) imply an algebraic asymptotic behavior, $S_{-+}(t) \simeq 4/\sqrt{\pi t}$ and $S(t) \simeq 2/\sqrt{\pi t}$ for $t \to \infty$. Thus, the domain decay exponent is $\psi = 1/2$.

To derive Eqs. (55) and (56) let us consider a sequence of interfaces starting from the right interface of our domain as a random walk. Namely, we set $W_0 = 0$, then define $W_1 = W_0 + v_1 = v_1$ where $v_1 = 1$ is the velocity of the right interface of our domain. We repeat this procedure so that $W_j = W_{j-1} + v_j$ and we treat W_j as the displacement of the random walk, started from the origin, at the $j^{\rm th}$ step. When the displacement becomes negative for the first time, the corresponding interface will move to the left and will eventually destroy the domain. Let we meet this interface after $2N+1$ steps. The corresponding probability P_N is readily determined by random walk methods [33]:

$$P_N = 2^{-2N}\,\frac{(2N)!}{(N+1)!N!}. \tag{57}$$

The same analysis applies to the left interface of the domain. Thus we have $2N+1$ interfaces to the right and $2M+1$ interfaces to the left. Our original domain survives till time t if the distance between the extreme interfaces is greater than $2t$. In other words, the interval of length $2t$ with the left boundary at the initial location of the extreme left interface should contain $2N+2M$ interfaces at most. The probability of this event is

$$\begin{aligned} U_{N+M}(t) &= e^{-2t}\sum_{k=0}^{N+M}\frac{(2t)^k}{k!} \\ &= \int_{2t}^{\infty} du\, e^{-u}\,\frac{u^{2N+2M}}{(2N+2M)!}. \end{aligned} \tag{58}$$

The survival probability $S_{-+}(t)$ is now given by

$$S_{-+}(t) = \sum_{N,M\geq 1} P_N P_M U_{N+M}(t). \tag{59}$$

It proves convenient to expand the summation in (59) to $N=0$ and $M=0$. This gives

$$\begin{aligned} S_{-+}(t) = &\sum_{N,M\geq 0} P_N P_M U_{N+M}(t) \\ &- 2\sum_{N\geq 0} P_N U_N(t) + U_0(t). \end{aligned} \tag{60}$$

The second sum in Eq. (60) can be rewritten as

$$\sum_{N=0}^{\infty} P_N U_N(t) = \int_{2t}^{\infty} du\, e^{-u}\sum_{N=0}^{\infty} P_N\,\frac{u^{2N}}{(2N)!}. \tag{61}$$

The sum on the right-hand side of Eq. (61) is compressed into $\sum_{N\geq 0} P_N\, u^{2N}/(2N)! = 2I_1(u)/u$, and the resulting integral is

$$\int_{2t}^{\infty} du\, e^{-u}\,\frac{I_1(u)}{u} = e^{-2t}[I_0(2t) + I_1(2t)]. \tag{62}$$

Thus $S_{-+}(t)$ becomes

$$\begin{aligned} S_{-+}(t) = &\int_{2t}^{\infty} du\, e^{-u}\sum_{N,M\geq 0} P_N P_M\,\frac{u^{2N+2M}}{(2N+2M)!} \\ &- 4e^{-2t}[I_0(2t) + I_1(2t)] + e^{-2t}. \end{aligned} \tag{63}$$

$S(t)$ is found from (52), (54), and (63) to yield

$$S(t) = \frac{1}{4}\int_{2t}^{\infty} du\, e^{-u} \sum_{N,M\geq 0} P_N P_M \,\frac{u^{2N+2M}}{(2N+2M)!}. \quad (64)$$

To perform the summation in Eq. (64) we need the combinatorial identity $\sum_{N+M=L} P_N P_M = 4P_{L+1}$ which can be checked directly. One can also establish this identity geometrically by noting that $2^{2N}P_N = (2N)!/N!(N+1)!$ gives the number of random walks starting at the origin and returning to the origin for the first time after $2N+2$ steps [33]. An appropriate counting of all such walks of length $2L+4$ then leads to the above identity. Making use of this identity we reduce Eq. (64) to

$$\begin{aligned} S(t) &= \frac{1}{4}\int_{2t}^{\infty} du\, e^{-u} \sum_{L=0}^{\infty} \left(\frac{u}{2}\right)^{2L} \frac{2(2L+1)}{L!(L+2)!} \\ &= 2\int_{2t}^{\infty} du\, e^{-u} \left[\frac{I_1(u)}{u} - \frac{3I_2(u)}{u^2}\right] \qquad (65) \\ &= \frac{1}{4}\int_{2t}^{\infty} du\, e^{-u} \left[I_0(u) - I_4(u)\right]. \end{aligned}$$

In deriving the second line we have used the definition of the modified Bessel functions; the third line has been derived by applying the identity [34] $I_{n-1}(u) - I_{n+1}(u) = \frac{2n}{u} I_n(u)$. Computing now the integral in the last line of Eq. (65) we arrive at (56). This completes the proof of Eqs. (55) and (56).

One can try to compute $Q_m(t)$, the domain number density. First we note that $Q_1(t)$ with specified boundary velocities can be readily found: $Q_{+-}(t) = e^{-2t}$, while other single-domain densities $Q_{++}(t) = Q_{--}(t)$ and $Q_{-+}(t)$ can be expressed via single-particle survival probabilities $S_\pm(t)$:

$$Q_{++}(t) = S_+(t), \quad Q_{-+}(t) = S_-(t)S_+(t). \qquad (66)$$

Then the total single-domain survival probability reads $Q_1(t) = [Q_{++}(t)+Q_{+-}(t)+Q_{-+}(t)+Q_{--}(t)]/4$. Asymptotically, $Q_1(t) \simeq (4\pi t)^{-1/2}$ implying exponents $\delta = \psi = \nu = 1/2$. Moreover the persistence exponent $\theta = 1$, and thus all exponents are identical with those of the $q = \infty$ Potts model. These two models exhibit several other similarities [44]. However, the present deterministic model of coarsening is quite different in that the number distribution $Q_m(t)$ is nontrivial. The determination of this distribution is more involved and left for the future.

IV. DISCUSSION AND SUMMARY

Even in one dimension there are many interesting situations where the above coarsening exponents are unknown. The simplest case is a diffusion equation which can describe coarsening in systems with non-conserved order parameter [1]. Recently, the persistence characteristics for the diffusion equation process have been investigated [19] numerically and theoretically by an approach close to the IIA. Given the enormous role played by the diffusion equation in science, surprisingly little is known about its underlying coarsening process [20].

Another well known coarsening process is the 1D time-dependent Ginzburg-Landau equation for a scalar non-conserved order parameter [1]. In this system, domains do not move and the coarsening proceeds via flipping of the shortest domains. The minimum size grows logarithmically [45], so it is convenient to define the coarsening exponents in terms of the minimum size L rather than time t. This process is solvable in that the domain size distribution, $P_n(L) = L^{-2}\mathcal{P}(n/L)$, is known [45,1]. The same expression holds for the domain number distribution. Some coarsening exponents are simple, $\nu = 1$ and $\delta = \sigma = \infty$. In contrast, the persistence exponent, $\theta \cong 0.1750758$ [4], is non-trivial. This process resembles the $q \to 1$ Potts model where $\psi = \theta$ as well.

It would be interesting to extend of our work to coarsening systems with *conserved* order parameter. Besides the dynamical exponent $z = 3$ little is known even for the one-dimensional Ising model with Kawasaki spin-exchange dynamics [46]. The coarsening exponents ψ and δ appear to be non-trivial for the Ising-Kawasaki model [47] as well. Another possible direction is to study the coarsening exponents for the natural generalization of the ballistic annihilation process, the N-species Lotka-Volterra process [15,48].

In summary, we introduced the domain size distribution and showed that it obeys scaling and is characterized by two independent nontrivial decay exponents. The survival probabilities of a domain and an unreacted domain are described by the exponents ψ and δ, respectively. Generally, these exponents obey $0 \leq \psi \leq \theta$ and $\psi \leq \nu \leq \delta$. In the most examples the above inequalities are strict; however, there are counter examples where $\psi = \theta$ and/or $\delta = \nu$. For the 1D $T = 0$ q-state Potts-Glauber model we developed the IIA that predicts the correct qualitative behavior of the domain size and number distributions and even reasonable estimates for the decay exponents. We also worked out the analytically tractable limits of $q \to 1$ and $q \to \infty$. It still remains, however, to obtain the exact behavior for general q. This might be possible using the techniques used in studies of single-spin persistence [6,8,12]. In a static version of the Potts model, an exact solution was presented and exponential decay of the domain density still occurred. It was also shown analytically that the coarsening exponents in the solvable *deterministic* ballistic annihilation model and the *stochastic* $q = \infty$ Potts model are identical.

These results indicate that several nontrivial decay laws underlie the evolution of elementary processes such as the non-equilibrium Ising model. These nontrivial exponents do not emerge naturally from studies of traditional quantities such as spatiotemporal correlations. It remains a challenge to find and obtain these underlying "hidden" exponents from a more systematic method. It is also intriguing to determine whether an entire hierarchy or a finite number of independent decay modes are present in these systems.

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