First Passage Properties of the Pólya Urn Process

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We study first passage statistics of the Pólya urn model. In this random process, the urn contains two types of balls. In each step, one ball is drawn randomly from the urn, and subsequently placed back into the urn together with an additional ball of the same type. We derive the probability G_n that the two types of balls are equal in number, for the first time, when there is a total of $2n$ balls. This first passage probability decays algebraically, $G_n \sim n^{-2}$, when n is large. We also derive the probability that a tie ever happens. This probability is between zero and one, so that a tie may occur in some realizations but not in others. The likelihood of a tie is appreciable only if the initial difference in the number balls is of the order of the square-root of the total number of balls.

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I. INTRODUCTION

Urn models play a central role in probability theory and combinatorics [1, 2]. Since the balls can represent anything from atoms to biological organisms to humans, urn models are widely used in the physical, life, and social sciences [3].

In this paper, we investigate the classic Pólya urn model [4–6]. This urn process is a type of birth process, and it is useful for modeling the spread of infectious diseases, population dynamics, and evolutionary processes in biology [5, 7–10]. Furthermore, this stochastic process is a branching process [11], and it is used to model data structures in computer science [12–14]. From the myriad of other applications, we mention decision making [15], reinforcement learning [16], and technology usage [17, 18]. We also note that the Pólya urn model is a limiting case of earlier urn models investigated by Laplace [19], Markov [20], and Ehrenfest [21].

The Pólya urn model exhibits rich and interesting phenomenology that includes strong influence of the initial conditions, large realization-to-realization fluctuations, and substantial finite-size corrections [6, 22, 23]. In this study, we obtain the first passage properties [24] of the Pólya urn model, and contrast these with the first passage characteristics of an ordinary random walk [24, 25].

In the Pólya urn model, there are two types of balls, black and white. In a basic step, one ball is selected randomly from all balls in the urn. This ball is then returned to the urn together with an additional ball of the same color. Starting with a given configuration of balls, the number of balls increases indefinitely by repeating this step ad infinitum. Thus, a configuration (B, W) with B black balls and W white balls evolves according to

$$(B, W) \to \begin{cases} (B+1, W) & \text{with probability} \quad \frac{B}{B+W}, \\ (B, W+1) & \text{with probability} \quad \frac{W}{B+W}. \end{cases} \tag{1}$$

We investigate first passage properties of the urn process (figure 1). We find that the probability G_n that a tie is reached, for the first time, when there are n balls of each type decays algebraically

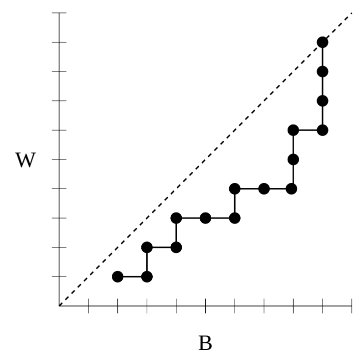

FIG. 1: The urn process as a trajectory on a two-dimensional lattice (bold line) where bullets indicate intermediate stages of the trajectory. Exit from the $B > W$ region is equivalent to this trajectory reaching the diagonal (broken line).

$$G_n \sim n^{-2}, \tag{2}$$

for large n. This asymptotic behavior holds for arbitrary initial conditions. We also show that the total exit probability, that is, the probability that a tie is ever reached, is less than one. Hence, an initial imbalance in the number of balls can be locked in forever. We study how the total exit probability depends on the initial condition and find that it is appreciable only when the imbalance in the number of balls is of the order of square-root of the total number of balls.

The rest of this paper is organized as follows. We derive the first passage probability in section II. We then obtain the exit probability by summing the first passage probability (section III). We discuss the extreme cases of nearly-maximal and extremely small exit probabilities (sec. IV), and then establish scaling properties of the exit probability (sec. V). We generalize the results to near-ties in section VI, and conclude in section VII.

II. THE FIRST PASSAGE PROBABILITY

Our goal is to quantify the first passage process, illustrated in figure 1, using the first passage probability and the total exit probability. For the initial condition $(B, W) = (b, w)$ where, without loss of generality, black balls are in the majority, $b > w$, the first passage probability $G_n(b, w)$ is the likelihood that a tie is reached, for the first time, when $(B, W) = (n, n)$. In other words $G_n(b, w)$ is the probability that the initial imbalance holds, $B > W$, if and only if $W < n$. The total exit probability $E(b, w)$ is the probability that a tie is ever reached.

These first passage characteristics are of interest in a variety of contexts. For example, in growth of bacterial colonies, when bacteria proliferate without resource limitations, cell death can be neglected, and the growth can be modeled by a branching process of two types [26, 27]. This process and the Polya urn model have the same exit probability, which measures the likelihood that the minority species eventually overtakes the majority species. In the context of binary search trees, first passage statistics quantify the likelihood that two branches of a tree reach perfect balance. Balanced trees are relevant in data structures, because they lead to more efficient searches. Indeed, the Polya urn process has been used to model such data structures in computer science [28].

As a preliminary step to finding the first passage probability, we obtain the likelihood that the system reaches configuration $(B, W) = (m, n)$ starting from $(B, W) = (b, w)$. Let's consider, for example, the transition $(1, 1) \to (3, 3)$ where one possible path is

$$(1, 1) \to (1, 2) \to (1, 3) \to (2, 3) \to (3, 3)\,.$$

The likelihood of this path is

$$\frac{1}{2} \times \frac{2}{3} \times \frac{1}{4} \times \frac{2}{5} = \frac{(1 \cdot 2) \cdot (1 \cdot 2)}{2 \cdot 3 \cdot 4 \cdot 5}\,. \qquad (3)$$

There are $\binom{4}{2} = 6$ distinct routes from $(1, 1)$ to $(3, 3)$ and they all have the same same probability $1/30$.

In general, all paths from configuration (b, w) to configuration (m, n) have the same probability

$$\frac{[b(b+1)\cdots(m-1)] \cdot [w(w+1)\cdots(n-1)]}{(b+w)(b+w+1)\cdots(m+n-1)}\,.$$

We rewrite this probability using factorials

$$\frac{(m-1)!}{(b-1)!} \times \frac{(n-1)!}{(w-1)!} \times \frac{(b+w-1)!}{(m+n-1)!}\,.$$

The total number of distinct paths from (b, w) to (m, n) equals the binomial $\binom{m+n-b-w}{m-b}$. Hence, the transition probability P that, starting from configuration (b, w), the system reaches configuration (m, n) is [3–6]

$$P = \binom{m-1}{b-1}\binom{n-1}{w-1}\binom{m+n-1}{b+w-1}^{-1}. \qquad (4)$$

In particular, the probability distribution is flat [4, 5] $P = \frac{1}{m+n-1}$, for the initial condition $(b, w) = (1, 1)$.

The number of paths from (b, w) to (n, n) that reach the diagonal $B = W$ only at the end point equals $\frac{b-w}{2n-b-w}$ times the total number of such paths. This result can be established using the reflection principle [1]. Since all paths from (b, w) to (n, n) are equiprobable, the first passage probability is simply $G_n(b, w) = \frac{b-w}{2n-b-w}P$. By substituting $m = n$ into Eq. (4), we obtain our first main result, the first passage probability

$$G_n(b, w) = \frac{b-w}{b+w}\binom{n-1}{b-1}\binom{n-1}{w-1}\binom{2n-1}{b+w}^{-1}. \qquad (5)$$

This quantity decays algebraically,

$$G_n(b, w) \simeq A(b, w)\, n^{-2}, \qquad (6)$$

in the asymptotic limit $n \gg b, w$. The proportionality constant in (6) is

$$A(b, w) = \frac{(b-w)(b+w-1)!}{(b-1)!(w-1)!}\, 2^{-b-w}\,.$$

For the special case $(b, w) = (2, 1)$, the first passage probability (5) is simply

$$G_n(2, 1) = \frac{1}{(2n-3)(2n-1)}. \qquad (7)$$

The probability that a tie is ever reached equals the sum $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}$, and hence, there is a finite chance that the initial imbalance in the number of balls is maintained forever. This behavior is different than that of a one-dimensional random walk. In an ordinary random walk, the two elementary transitions in (1) occur with probability $1/2$, and the first passage probability decays algebraically, $G_n \sim n^{-3/2}$, for large n. Yet, the exit probability equals one, and the random walk is guaranteed to reach the diagonal $B = W$. Thus, the one-dimensional random walk is recurrent, but the Pólya urn process is transient.

III. THE EXIT PROBABILITY

The exit probability $\mathcal{E}_n(b, w)$ is the likelihood that starting from configuration (b, w), a tie happens by the time the urn contains $2n$ balls. The exit probability follows from the first passage probability,

$$\mathcal{E}_n(b, w) = \sum_{b \le j \le n} G_j(b, w)\,. \qquad (8)$$

The lower limit reflects that the quickest tie occurs when $(B, W) = (b, b)$. We are especially interested in the total exit probability, $E(b, w) \equiv \lim_{n\to\infty} \mathcal{E}_n(b, w)$. From the identity $\mathcal{E}_n - \mathcal{E}_{n-1}(b, w) = G_n(b, w)$ and equation (6), we conclude the asymptotic behavior

$$E(b, w) - \mathcal{E}_n(b, w) \simeq A(b, w)\, n^{-1}, \qquad (9)$$

when $n \gg b, w$. In particular, the quantity $\mathcal{E}_n(2,1) = \frac{n-1}{2n-1}$ that is the sum of (7), agrees with (9).

To evaluate the total exit probability $E(b,w)$, we introduce the shorthand notation $C_k(b,w) \equiv G_{b+k}(b,w)$. With this notation, Eq. (8) becomes $E(b,w) = \sum_{k\geq 0} C_k(b,w)$, and

$$C_k(b,w) = \frac{b-w}{b+w} \frac{\binom{b+k-1}{b-1}\binom{b+k-1}{w-1}}{\binom{2b+2k-1}{b+w}}, \qquad (10)$$

which is obtained by substituting $n = b+k$ into (5). In particular, the quantity

$$C_0(b,w) = \frac{\Gamma(b)\,\Gamma(b+w)}{\Gamma(2b)\,\Gamma(w)}$$

is the probability that a tie occurs as quickly as possible.

In terms of the quantities $C_k(b,w)$, the total exit probability $E(b,w) = \sum_{j\geq b} G_j(b,w)$ equals

$$E(b,w) = \sum_{k\geq 0} C_k(b,w). \qquad (11)$$

We now evaluate the ratio of two consecutive first passage probabilities

$$\frac{C_{k+1}(b,w)}{C_k(b,w)} = \frac{(k+b)(k+\frac{b-w}{2})(k+\frac{b-w+1}{2})}{(k+1)(k+b+\frac{1}{2})(k+b-w+1)}. \qquad (12)$$

Given these ratios, the exit probability can be expressed in terms of the hypergeometric function [29]

$$E(b,w) = \tfrac{\Gamma(b)\Gamma(b+w)}{\Gamma(2b)\Gamma(w)} F(b, \tfrac{b-w}{2}, \tfrac{b-w+1}{2}; b+\tfrac{1}{2}, b-w+1; 1). \qquad (13)$$

This closed form expression is our second main result. As expected, $E(b,b) = 1$ and $E(b,0) = 0$.

We also note that the exit probability satisfies the compact recursion relation

$$E(b,w) = \frac{b}{b+w}\, E(b+1,w) + \frac{w}{b+w}\, E(b,w+1), \qquad (14)$$

for all $b \neq w$. The boundary conditions are $E(b,b) = 1$ and $E(b,0) = 0$. This recursion follows directly from the definition of the stochastic process (1), and is reminiscent of the recursion equation for an ordinary random walk $E(b,w) = \frac{1}{2}\, E(b+1,w) + \frac{1}{2}\, E(b,w+1)$ [24]. We use this recursion to analyze extremal and scaling properties of $E(b,w)$ in the next section.

IV. EXTREMAL BEHAVIOR

Intuitively, we expect that when b is fixed, the exit probability increases monotonically with w. The exit probability is largest when the number of balls is balanced, $E(b,b) = 1$, and conversely, the exit probability is smallest when the initial imbalance is maximal, $E(b,0) = 0$. We now discuss the extreme cases of very small and nearly-maximal exit probabilities, respectively.

When $w = 1$, the exit probability decays exponentially with the total number of balls,

$$E(b,1) = 2^{1-b}. \qquad (15)$$

To obtain this result, we note that when $w = 1$, two of the arguments of the hypergeometric function in (13) coincide and hence, $E(b,1) = \frac{\Gamma(b)\,\Gamma(b+1)}{\Gamma(2b)}\, F\left(\frac{b}{2}, \frac{b-1}{2}; b+\frac{1}{2}; 1\right)$. We obtain the expression (15) using the Gauss identity for the hypergeometric function

$$F(x,y;z;1) = \frac{\Gamma(z-x-y)\,\Gamma(z)}{\Gamma(z-x)\,\Gamma(z-y)}, \qquad (16)$$

and the following two identities for the Gamma function, $\Gamma(x+1) = x\Gamma(x)$, and $\Gamma(\frac{1}{2})\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)$.

By substituting $E(b,1) = 2^{1-b}$ into the recursion (14), we have $E(b,2) = (b+2)2^{-b}$. Similarly, we obtain $E(b,3) = (b^2+5b+8)2^{-b-2}$ by substituting $E(b,2)$ into (14). In general, the exit probability has the form

$$E(b,w) = \frac{U_w(b)}{(w-1)!}\, 2^{2-w-b}, \qquad (17)$$

where $U_w(b)$ is a polynomial of degree $w-1$ in the variable b. From equation (14), these polynomials satisfy the recursion

$$U_{w+1}(b) = 2(b+w)U_w(b) - b\, U_w(b+1). \qquad (18)$$

Starting with the boundary condition, $U_1(b) = 1$, we have

$$U_w(b) = \begin{cases} 1 & w=1, \\ b+2 & w=2, \\ b^2+5b+8 & w=3, \\ b^3+9b^2+32b+48 & w=4, \\ b^4+14b^3+83b^2+262b+384 & w=5. \end{cases} \qquad (19)$$

Since the coefficient of the dominant term in $U_w(b)$ equals one, the exit probability decays exponentially with the total initial population

$$E(b,w) \simeq \frac{b^{w-1}}{(w-1)!} 2^{2-b-w}, \qquad (20)$$

when w is finite and $b \to \infty$.

To analyze the behavior in the opposite limit of nearly-maximal exit probabilities, we consider the special case $w = b-1$ where the ratio (12) simplifies as follows

$$\frac{C_{k+1}}{C_k} = \frac{(k+b)(k+\frac{1}{2})}{(k+2)(k+b+\frac{1}{2})}. \qquad (21)$$

We now shift the index of the first passage probability by one, $D_{k+1} \equiv C_k$, with $D_0 = -1$, and then evaluate the sum (11) to find $E(b,b-1) = 1 - F\left(b-1, -\frac{1}{2}; b-\frac{1}{2}; 1\right)$.

Further, we express the exit probability through Gamma functions by using the identity (16),

$$E(b, b-1) = 1 - \frac{\Gamma(b-1/2)}{\Gamma(b)\Gamma(1/2)}. \tag{22}$$

The exit probability increases monotonically with b: $E(b, b-1) = 1/2, 5/8, 11/16$ for $b = 2, 3, 4$. Moreover, ties become practically certain, $E(b, b-1) \simeq 1 - 1/\sqrt{\pi b}$, in the limit $b \to \infty$.

Along the same lines, we evaluate $E(b, b-q)$ by substituting the form (22) and the boundary condition $E(b, b) = 1$ into the recursion (14),

$$E(b, b-q) = 1 - \frac{\Gamma(b-1/2)}{\Gamma(b)\Gamma(1/2)} \times \begin{cases} 1 & q = 1, \\ 2 & q = 2, \\ 3\frac{b-5/3}{b-3/2} & q = 3, \\ 4\frac{b-2}{b-3/2} & q = 4. \end{cases} \tag{23}$$

From these examples, we conclude $E(b, b-q) \simeq 1 - q/\sqrt{\pi b}$ when q is finite and $b \to \infty$. In other words,

$$E(b, w) \simeq 1 - \sqrt{\frac{2}{\pi}} \frac{|b-w|}{\sqrt{b+w}}, \tag{24}$$

when the initial imbalance $|b - w|$ is fixed and the total number of balls $b + w$ diverges. We used the symmetry $E(x, y) = E(y, x)$ so that (24) applies for both $b > w$ and $w > b$. This equation implies that a tie is nearly certain whenever the initial discrepancy in the number of balls is much smaller than the square-root of the total number of balls. Otherwise, the exit probability is substantially reduced.

V. TYPICAL BEHAVIOR

The asymptotic behavior (24) suggests that the exit probability is function of a single variable when the total number of balls is very large. Specifically, the scaling function $\Phi(z)$, given by

$$E(b, w) \simeq \Phi(z) \quad \text{with} \quad z = \frac{|b-w|}{\sqrt{b+w}}, \tag{25}$$

quantifies the exit probability in the limit $b + w \to \infty$.

To show this scaling behavior, we make the change of variables $(b, w) \to (z, N)$ with the scaling variable $z = (b-w)/\sqrt{b+w}$ and the time-like variable $N = b+w$. Here, the case $b > w$ is considered without loss of generality. We now consider the limit of very large N and finite z, substitute $E(b, w) = \tilde{\Phi}(z, N)$ into the recursion equation (14), and expand for very large N. Specifically, we substitute $b = \big(N + z\sqrt{N}\big)/2$ and $w = \big(N - z\sqrt{N}\big)/2$ and the corresponding expressions for $b + 1$ and $w + 1$ into (14), and identify the leading terms in the limit $N \to \infty$. This analysis shows that the scaling function

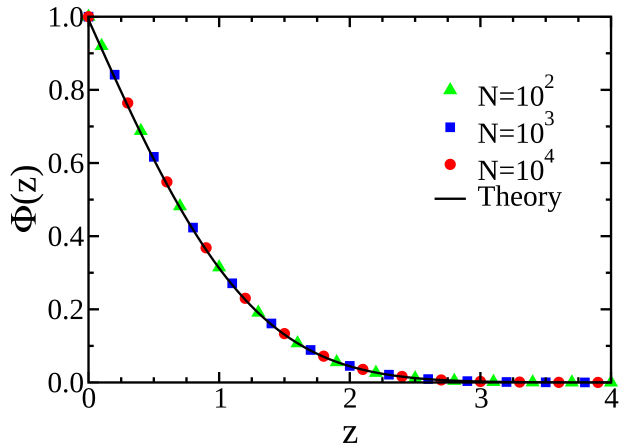

FIG. 2: The scaling function $\Phi(z)$ given in (27) versus the scaling variable z (solid line). Also shown is the exit probability $E(b, w)$, given by the exact expression (13), versus z for three different values of $N = b + w$ (symbols).

$\Phi(z) \equiv \lim_{N\to\infty} \tilde{\Phi}(z, N)$ satisfies the differential equation

$$\frac{d^2\Phi(z)}{dz^2} + z\,\frac{d\Phi(z)}{dz} = 0. \tag{26}$$

The boundary conditions are $\Phi(0) = 1$ and $\Phi(\infty) = 0$. The solution to (26) subject to these boundary conditions is simply

$$\Phi(z) = \text{erfc}\left(\frac{z}{\sqrt{2}}\right), \tag{27}$$

where $\text{erfc}(z)$ is the complementary error function [29], $\text{erfc}(x) = (2/\sqrt{\pi}\,) \int_x^\infty du \exp(-u^2)$. The scaling function is monotonically decreasing because a larger initial imbalance implies a smaller exit probability (figure 2). The small-z and large-z behaviors of the scaling function are as follows,

$$\Phi(z) \simeq \begin{cases} 1 - \sqrt{2/\pi}\; z & z \ll 1, \\ \sqrt{2/\pi}\, z^{-1} \exp(-z^2/2) & z \gg 1. \end{cases} \tag{28}$$

The small argument behavior agrees with (24), while the large argument behavior is consistent with (15). Indeed, when $w = 1$ and b is very large, the scaling variable is $z \simeq \sqrt{b}$, and hence, $\ln \Phi \sim -z^2 \sim -b$.

The scaling form (25) implies that the likelihood of a tie is appreciable only when the initial population difference, $\Delta = |b - w|$, is of the same order as the square-root of the total population, $N = b + w$, that is,

$$\Delta \sim \sqrt{N}. \tag{29}$$

A tie is nearly certain when the discrepancy is small, $\Delta \ll \sqrt{N}$, but extremely rare when $\Delta \gg \sqrt{N}$.

VI. NEAR TIES

There are a number of generalizations of the first passage process discussed above. Two natural questions are: (i) what is the probability that the *ratio* between the majority population and the majority population is always above a fixed threshold and (ii) what is the probability that the *difference* between the two populations is always above a fixed threshold. In this section, we address the latter problem.

We define $G_n(b,w;d)$ to be the first passage probability that starting with configuration (b,w), the difference $B-W>d$ if and only if $W<n$. In other words $G_n(b,w;d)$ is the probability that there are at least d more black balls throughout the evolution and moreover, this condition is violated for the first time when $(B,W)=(n+d,n)$. We obtain the first passage probability

$$G_n(b,w;d) = \frac{b-w-d}{b+w}\frac{\binom{n+d-1}{b-1}\binom{n-1}{w-1}}{\binom{2n+d-1}{b+w}}, \tag{30}$$

by multiplying the probability (4) for transitioning from (b,w) to $(n+d,n)$ with the fraction $\frac{b-w-d}{2n+d-b-w}$ of these paths that do not cross the line $B=W+d$ [1]. Of course, this expression matches (5) when $d=0$. Again, the asymptotic behavior in the limit $n\to\infty$ is $G_n\sim n^{-2}$.

The exit probability $E(b,w;d)$ is the probability that the line $B=W+d$ is reached at least once during the evolution. By repeating the steps leading to (13), we find the exit probability in terms of a higher-order hypergeometric function

$$E(b,w;d) = \tfrac{\Gamma(b-d)\,\Gamma(b+w)}{\Gamma(2b-d)\,\Gamma(w)}\,F(c_1,c_2,c_3,c_4;e_1,e_2,e_3;1). \tag{31}$$

The corresponding arguments are

$$\begin{gathered} c_1=b,\quad c_2=\tfrac{b-w-d}{2},\quad c_3=\tfrac{b-w-d+1}{2},\quad c_4=b-d,\\ e_1=b+\tfrac{1-d}{2},\quad e_2=b-w-d+1,\quad e_3=b-\tfrac{d}{2}. \end{gathered}$$

For near-ties, $d=1$, there are many similarities with $E(b,w)$ of Eq. (13). For example, the exit probability decays exponentially when the initial imbalance is maximal,

$$E(b,1;1)=\frac{b}{b-1}2^{1-b}. \tag{32}$$

Additionally, the exit probability is close to one when the initial imbalance is minimal,

$$E(b,b-2;1) = 1-\frac{1}{2(b-1)}-\frac{\Gamma(b-1/2)}{\Gamma(b)\Gamma(1/2)}. \tag{33}$$

Thus, the behavior $E(b,b-2;1)\simeq 1-1/\sqrt{\pi\, b}$ is recovered when $b\to\infty$.

VII. DISCUSSION

In summary, we obtained first passage characteristics of the Pólya urn process as a function of the initial condition. The first passage probability that a tie is reached for the first time when there are $2n$ balls decays algebraically, $G_n\sim n^{-2}$, for large n. The probability that a tie ever occurs, is less than one, hence ties are not certain. This exit probability decreases as the initial discrepancy in the number of balls increases. Moreover, there is a universal scaling behavior when the total initial population is very large. This scaling behavior implies that the exit probability is appreciable only when the initial population imbalance is of the order of the square-root of the total population.

The key property of the Pólya urn model is that the fraction of white balls approaches a limiting value, but this value fluctuates from realization-to-realization. In many other urn models, however, the opposite is true, and moreover, the two fractions approach the same limiting value. This is the case for the Friedman urn process [30, 31] which, in its simplest form, is equivalent to the stochastic process

$$(B,W)\to\begin{cases}(B+1,W) & \text{with probability}\quad \frac{W}{B+W},\\ (B,W+1) & \text{with probability}\quad \frac{B}{B+W}.\end{cases}$$

It will be interesting to investigate first passage properties of this urn process. We conjecture that first passage statistics are much closer to those of the ordinary random walk.

In the context of population dynamics and evolutionary biology, continuous time processes are more appropriate [11]. The continuous time analog of the Pólya urn model (1) is the two-species branching process, $B\to B+B$, and $W\to W+W$, where the two birth rates are equal. This continuous time process is closely related to the discrete time urn process. For instance, our results for the total exit probabilities are valid for the continuous time process as well.

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