Multiscaling in fragmentation

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We review recent results on random fragmentation of multidimensional objects. In addition to the obvious volume conservation, these processes exhibit an infinite number of hidden conservation laws. The fragment size distribution is characterized by an infinite number of scales and thus, exhibits multiscaling. Nevertheless, the volume distribution function shows ordinary scaling, *i.e.*, it is characterized by a single scale.

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Fragmentation underlies a number of physical, chemical, and geological processes, such as polymer degradation, atomic collisions cascades, energy cascades in turbulence, martensitic transformations, multivalley structures of the phase space of disorder systems, meteor impacts, *etc.* [1-8] The fragmented quantity in such processes are diverse: mass, momentum, energy, or area. A characteristic feature of these cascade processes is that fragments continue splitting independently. One simplifying assumption used in most theoretical studies is that fragments may be described properly by a single variable, say their mass or size [9-15]. However, in many cases, more than one variable may be necessary. Below, we review recent results concerning fragmentation with multiple variables [16-22]. We generalize the random scission model to higher dimension and solve it analytically. Using the moments of the fragment size distribution, we show that the process exhibits many interesting properties such as multiscaling and the existence of an infinite number of hidden conservation laws. We also discuss applications of the model to arbitrary dimensions, homogeneous breakage kernels, and stochastic fractals.

The random scission model [5,9,10] is the simplest realization of a fragmentation process, where the distribution of fragments of length x, $P(x,t)$, satisfies the integro-differential equation

$$\frac{\partial P(x,t)}{\partial t} = -xP(x,t) + 2\int_x^\infty dy P(y,t). \qquad (1)$$

The negative term on the right-hand side represents loss due to binary breakups. Since fragmentation is uniform, the corresponding loss rate is proportional to the fragment size. The gain term accounts for the increase of fragments of size x due to breakups of longer fragments. The asymptotic solution to equation (1) can be written in a scaling form

$$P(x,t) \simeq \frac{C}{\langle x\rangle^2}\Phi\left(\frac{x}{\langle x\rangle}\right), \qquad (2)$$

with $\langle x\rangle \simeq t^{-1}$ and the scaling function $\Phi(z) = e^{-z}$. The constant $C = \int dx P(x,0)$ ensures conservation of the total mass. The random scission process is equivalent to uniform deposition of points on a one-dimensional interval. Indeed, every deposition of such a "crack" leads to fragmentation of the underlying segment.

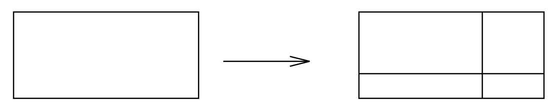

FIG. 1. The fragmentation process.

The following process is a natural generalization to two dimensions: a fragmentation event takes place at a random internal point of the rectangle and gives birth to four smaller rectangles as illustrated in Figure 1. The distribution function $P(x_1,x_2;t)$ describing rectangles of size $x_1\times x_2$ is governed by the following linear rate equation [16-19]

$$\frac{\partial P(x_1,x_2;t)}{\partial t} = -x_1x_2P(x,t) + 4\int_{x_1}^\infty\int_{x_2}^\infty dy_1dy_2P(y_1,y_2;t). \qquad (3)$$

A simple integration of this equation shows that the total area is conserved, $\int_0^\infty\int_0^\infty dx_1dx_2x_1x_2P(x_1,x_2;t) =$ const. The average number of fragments $N(t) = N(0)+3t$ is readily found. The rate of creation of rectangles reflects the fact that 3 additional rectangle are created in each fragmentation event. It is useful to introduce the double Mellin transform (or alternatively the moments) of the distribution function $P(x_1,x_2;t)$,

$$M(s_1,s_2;t) = \int_0^\infty\int_0^\infty dx_1dx_2x_1^{s_1-1}x_2^{s_2-1}P(x_1,x_2;t) \qquad (4)$$

The rate equation (3) implies the following rate equation for the moments,

$$\frac{\partial M(s_1,s_2;t)}{\partial t} = \left(\frac{4}{s_1s_2}-1\right)M(s_1+1,s_2+1;t). \qquad (5)$$

A surprising feature of Eq. (5) is that it implies the existence of an infinite number of conservation laws. The moments $M(s_1^*,s_2^*;t)$ with s_1^* and s_2^* satisfying $s_1^*s_2^*=4$

are time independent. Thus, in addition to the conservation of the total area $A \equiv M(2,2;t)$, there are an infinite number of hidden conserved integrals. These integrals are in fact responsible for the absence of scaling solutions to Eq. (3). Indeed, the scaling solution $P(x_1,x_2;t)=t^wQ(t^zx_1,t^zx_2)$, implies an infinite amount of scaling relations, $w=z(s_1^*+s_2^*)$ at $s_1^*s_2^*=4$, which cannot be satisfied by the scaling exponents, w and z.

Although an exact solution for the moment in terms of generalized hypergeometric functions is possible [17-19], we present an alternative technique for obtaining the asymptotic behavior of the moments [17]. Asymptotically, the moments depend algebraically on time

$$M(s_1,s_2;t)\simeq A(s_1,s_2)t^{-\alpha(s_1,s_2)}. \tag{6}$$

Substituting this form into Eq. (5) gives the difference equations $\alpha(s_1,s_2)+1=\alpha(s_1+1,s_2+1)$. Additionally, the conservation laws imply that $\alpha(s_1^*,s_2^*)=0$ when $s_1^*s_2^*=4$. Solving the difference equation subject to the boundary constraint gives $\alpha(s_1^*+k,s_2^*+k)=k$. Thus, the exponent α can be easily obtained by solving a quadratic equation:

$$\alpha(s_1,s_2)=\frac{1}{2}\left[s_1+s_2-\sqrt{(s_1-s_2)^2+16}\right]. \tag{7}$$

To see the nontrivial structure of the moments, consider the average value of $x_1^{n_1}x_2^{n_2}$ defined by

$$\langle x_1^{n_1}x_2^{n_2}\rangle=\frac{\int_0^\infty\int_0^\infty dx_1dx_2x_1^{n_1}x_2^{n_2}P(x_1,x_2;t)}{\int_0^\infty\int_0^\infty dx_1dx_2P(x_1,x_2;t)} \tag{8}$$

or equivalently, by $\langle x_1^{n_1}x_2^{n_2}\rangle = M(n_1+1,n_2+1;t)/M(1,1;t)$. For example, the ratio $\langle x_1^nx_2^n\rangle/\langle x_1^n\rangle\langle x_2^n\rangle\sim t^{\sqrt{n^2+16}-4}$ is time dependent, while for a scaling distribution such a ratio should approach a constant. Furthermore, the moments of the length $\langle l^n\rangle\equiv\langle x_1^n\rangle\equiv\langle x_2^n\rangle$ can be found as well

$$\langle l^n\rangle\sim t^{-(n+4-\sqrt{n^2+16})/2}. \tag{9}$$

Only in the limit $n\to 0$, the leading behavior $\langle l^n\rangle\sim t^{-n/2}$ follows an ordinary scaling behavior, as the exponent $n/2$ is linear in n. The general n dependence is more complicated. For example, $\langle l\rangle\sim t^{-(5-\sqrt{17})/2}\sim t^{-.438}$, and $\langle l^2\rangle\sim t^{-(3-\sqrt{5})}\sim t^{-.764}$. Note that the average length decays slower than the square root of the average area, $\sqrt{\langle x_1x_2\rangle}\sim t^{-1/2}$. We observe that the distribution function $P(x_1,x_2;t)$ in the two-dimensional random scission model does not approach a scaling form in the long-time limit. However, since all the moments still show a power-law behavior, we conclude that the model exhibits a multiscaling asymptotic behavior.

The moments provide an almost complete analytical description of the fragmentation process. However, a snapshot of the system at the later stages remains intriguing (see Fig. 2). This unexpectedly rich pattern arising in such a simple process can be viewed as a consequence of the fact that the process is not fully self-similar. Instead, the pattern is formed of sets of different scales which are spatially interwoven. Fig. 2 also shows that a number of rectangles have large aspect ratio. Indeed, the $n^{\rm th}$ moment of the aspect ratio,

$$\langle (x_1/x_2)^n\rangle\sim t^{\sqrt{n^2+4}-2}\qquad |n|<1, \tag{10}$$

is a growing function of time. In other words, rectangles tend to break into thin and long rectangles.

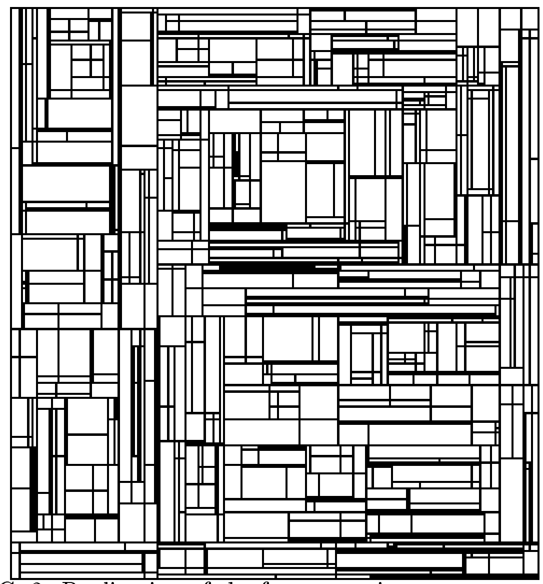

FIG. 2. Realization of the fragmentation process on a unit square at $t=1000$

Let us consider the *area* distribution function, $P(A,t)$, with $P(A,t)=\int_0^\infty\int_0^\infty dx_1dx_2\delta(x_1x_2-A)P(x_1,x_2;t)$. The corresponding "diagonal" moments $M(s;t)\equiv M(s,s;t)$ can be evaluated from the general solution $M(s,t)\sim t^{2-s}$. Consequently, $\langle A^n\rangle\sim\langle A\rangle^n\sim t^{-n}$, indicating that the area distribution function reaches a scaling form asymptotically,

$$P(A,t)\simeq t^2\Phi_2(At). \tag{11}$$

The exact scaling function is given by $\Phi_2(z)=6\int_0^1 d\zeta\left(\frac{1}{\zeta}-1\right)e^{-z/\zeta}$, and has the following limiting behaviors

$$\Phi_2(z)\to\begin{cases}6z^{-2}e^{-z}, & \text{if } z\gg 1,\\ 6\ln(1/z), & \text{if } z\ll 1.\end{cases} \tag{12}$$

Similar to the random scission model, large areas are exponentially suppressed. However, in contrast with the one-dimensional case, there is a weak logarithmic cusp in the limit of small areas.

The above results can be easily generalized to d dimensions. In the generalized version of the random scission model, each fragmentation event results in 2^d fragments. We denote the moments by $M(\mathbf{s};t)$ with

$\mathbf{s} \equiv (s_1, \ldots, s_n)$ and assume a power-law dependence, $M(\mathbf{s};t) \sim t^{-\alpha(\mathbf{s})}$. The exponents satisfy the recursion relation $\alpha(\mathbf{s}) + 1 = \alpha(\mathbf{s} + \mathbf{1})$, with $\mathbf{1} = (1, \ldots, 1)$. Meanwhile, the exponents should also reflect the hidden conserved integrals, *i.e.*, $\alpha(\mathbf{s}^*) = 0$ on the hypersurface $\mathbf{s}^*$, where $\Pi_j s_j^* = 2^d$. The solution to the recursion relations satisfying the boundary conditions is given by the formal expression

$$\alpha(\mathbf{s}) = \alpha(\mathbf{s}^* + k\mathbf{1}) = k. \tag{13}$$

Hence, the problem is reduced to finding roots of the algebraic equation $\Pi_j[s_j - \alpha(\mathbf{s})] = 2^d$. Since this equation is of degree d, an analytic solution is feasible only for $d \leq 4$.

All features found for the two-dimensional case such as multiscaling occur for higher dimensions as well. For example, nonuniversal behavior is found for the various moments of the length distribution. In the limit $d \gg 1$, we find that the moments decay asymptotically according to

$$\langle l^n \rangle \sim t^{-2\ln(1+n/2)/d}. \tag{14}$$

The n dependence is logarithmic in contrast to the linear dependence expected for scaling distributions. The diagonal exponents $\alpha(s, \ldots, s) = 2 - s$ indicate that the volume distribution exhibits scaling $P(V,t) \simeq t^2 \Phi_d(Vt)$. The limiting behavior of the scaling function can be evaluated as well

$$\Phi_d(z) \sim \begin{cases} z^{-2}e^{-z}, & z \gg 1, \\ \ln^{d-1}(1/z), & z \ll 1. \end{cases} \tag{15}$$

While the power law behavior is general, the logarithmic cusp occurs for $d > 1$. We conclude that the volume distribution generally obeys scaling, while for $d > 1$ other geometric characteristics such as the length, the surface area, *etc.* obey multiscaling.

One can also consider shape-dependent fragmentation rates and study homogeneous kernels $K = \mathbf{x}^{\mathbf{m}} \equiv x_1^{m_1} \cdots x_d^{m_d}$ (the case $\mathbf{m} = \mathbf{1}$ corresponds to the random scission model). When $m_i > 0$, this generalization also results in multiscaling of the fragment distribution. The moments $M(\mathbf{s}^*)$ with $\Pi_j(s_j^* + m_j - 1) = 2^d$ are conserved. Asymptotically, moments depend algebraically upon time, $M(\mathbf{s};t) \sim t^{-\alpha(\mathbf{s})}$, and the exponents can be found by solving the equation $\Pi_j[s_j^* + m_j - 1 - \alpha(\mathbf{s})] = 2^d$. The kernel K plays the role of the volume V as it exhibits regular scaling, $P(K,t) = t^2\Phi(Kt)$, or equivalently, $\langle K^n \rangle \sim \langle K \rangle^n \sim t^{-n}$. Of course, other geometric properties show multiscaling.

The special situation where the kernel is a homogeneous function of the volume $K = V^\lambda$ is of particular interest. The total number of fragments $N(t) \equiv M(\mathbf{1};t)$ grows according to $N(t) \sim t^{1/\lambda}$. Hence, the case $\lambda = 0$ is critical, and the number of fragments grows exponentially in time. Finally, for $\lambda < 0$, the shattering transition takes place: the total volume decreases monotonically and the total number of fragments reaches infinity within an infinitesimally small time interval. Moreover, a finite fraction of the volume breaks into zero-volume rectangles. This phenomenon is well known in the context of one-dimensional fragmentation [11,12] and has been examined in the context of two-dimensional fragmentation as well [19].

The above techniques are useful in a number of problems. Surprisingly, one can map random sequential adsorption of needles in 2D to a multifragmentation problem. Using a rate equation similar to equation (3), the needle density is found $n(t) \sim t^{\sqrt{2}-1}$ [16]. A recently suggested model for martensitic transformations can also be reduced to a two-dimensional fragmentation process where only two fragments are created at each event. Although the rate equations are different, the moment technique is still applicable and the resulting behavior is similar: the length multiscales, while the area scales [4,20]. Stochastic counterparts of canonical fractals such as the Cantor set and the Serpinsky gasket can also be considered using multifragmentation techniques, and one can obtain expressions for the fractal dimension and other multifractal properties [21,22].

In conclusion, fragmentation processes in spatial dimensions larger than one exhibit multiscaling in the long-time limit. Specifically, the length distribution function has moments that scale algebraically in time with an infinite number of independent length scales, while the area distribution function is characterized by a single length scale. The volume distribution function also exhibits a weak logarithmic singularity near the origin.

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