Nontrivial Velocity Distributions in Inelastic Gases

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We study spatially homogeneous inelastic gases using the Boltzmann equation. We consider uniform collision rates and obtain analytical results valid for arbitrary spatial dimension d and arbitrary dissipation coefficient ϵ. In the unforced case, we find that the velocity distribution decays algebraically, $P(v,t)\sim v^{-\sigma}$, for sufficiently large velocities. The exponent $\sigma(d,\epsilon)$ exhibits nontrivial dependence on the spatial dimension and the dissipation coefficient.

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Granular gases present novel challenges, previously not encountered in fluid dynamics [1]. Specifically, the strong underlying energy dissipation leads to clustering instabilities and strong velocity correlations [2–7]. A series of recent experimental and theoretical studies reveals a rich phenomenology. In particular, velocities are characterized by anomalous statistics, sensitive to the details of the driving conditions, the density, and the degree of dissipation [8–12].

Kinetic theory provides a systematic framework for deriving macroscopic properties from the microscopic collision dynamics [13–15]. Yet, analysis of the corresponding Boltzmann equation often involves uncontrolled approximations or use of nearly Maxwellian distributions. Motivated by the latter issue, we examine both unforced and forced inelastic gases using a simplified Boltzmann equation. Specifically, we employ Maxwell's collision rate which is proportional to the typical velocity rather than the relative velocity [16]. The resulting Boltzmann equation is analytically tractable, as reported in a few recent studies [17–19].

This kinetic theory leads to interesting behaviors in the freely evolving case. In one dimension, while moments of the velocity distribution exhibit multiscaling [17], the velocity distribution itself still approaches a scaling form with an algebraic large velocity tail [18]. An algebraic tail was also found numerically in two dimensions [18]. Here, we show analytically that in arbitrary spatial dimension the velocity distribution admits a scaling solution with an algebraic large velocity tail. The corresponding exponent, a root of a transcendental equation, depends on the spatial dimension and the restitution coefficient. In the driven case, we find that the velocity distribution is non-Maxwellian.

Our starting point is a homogeneous system of identical inelastic spherical particles in arbitrary spatial dimension d. The mass and the cross-section are set to unity without loss of generality. When two particles collide, the normal component of the relative velocity is reduced by the restitution coefficient $r=1-2\epsilon$, while the tangential component remains the same. Denoting by $\mathbf{n}$ the impact direction, the unit vector connecting the centers of the two particles, the post-collision velocities $\mathbf{v}_{1,2}$ are given by a linear combination of the precollision velocities $\mathbf{u}_{1,2}$,

$$\mathbf{v}_{1,2}=\mathbf{u}_{1,2}\mp(1-\epsilon)\,(\mathbf{g}\cdot\mathbf{n})\,\mathbf{n}, \tag{1}$$

with the relative velocity $\mathbf{g}=\mathbf{u}_1-\mathbf{u}_2$. The energy dissipated in a collision equals $\Delta E=-\epsilon(1-\epsilon)(\mathbf{g}\cdot\mathbf{n})^2$. When $\epsilon=0$ collisions are elastic, while when $\epsilon=1/2$ collisions are perfectly inelastic with maximal energy dissipation.

Let $P(\mathbf{v},t)$ be the normalized density of particles with velocity $\mathbf{v}$ at time t. Assuming molecular chaos or perfect mixing, we arrive at the following Boltzmann equation for the velocity distribution

$$\frac{\partial P(\mathbf{v},t)}{\partial t}=\langle g\rangle\int d\mathbf{n}\int d\mathbf{u}_1\int d\mathbf{u}_2\;P(\mathbf{u}_1,t)\,P(\mathbf{u}_2,t)\times \Big\{\delta\big[\mathbf{v}-\mathbf{u}_1+(1-\epsilon)(\mathbf{g}\cdot\mathbf{n})\,\mathbf{n}\big]-\delta(\mathbf{v}-\mathbf{u}_1)\Big\}. \tag{2}$$

The collision rate in the Maxwell approximation represents the typical velocity scale, $\langle g\rangle=\sqrt{T}$, with $T=\frac{1}{d}\langle v^2\rangle$ the "granular temperature" or the average kinetic energy per degree of freedom (for hard spheres, the collision rate equals the actual relative velocity $\mathbf{g}\cdot\mathbf{n}$). This evolution equation naturally describes a stochastic process where randomly chosen pairs of particles undergo inelastic collisions according to (1) with a randomly chosen impact direction $\mathbf{n}$. In writing Eq. (2) we tacitly ignored the restriction $\mathbf{g}\cdot\mathbf{n}>0$ on the integration range, because the integrand obeys the reflection symmetry $\mathbf{n}\to-\mathbf{n}$. The integration measure should be normalized, $\int d\mathbf{n}=1$.

In the absence of energy input the system "cools" indefinitely according to Haff's law [20]. Indeed, the time dependence of the temperature is found from the Boltzmann equation (2) to give $\frac{d}{dt}T=-\lambda T^{3/2}$ with $\lambda=2\epsilon(1-\epsilon)\int d\mathbf{n}\,n_1^2$ (the first axis was conveniently chosen to be parallel to $\mathbf{g}$). Since $n_1^2+\ldots+n_d^2=1$ and $\int d\mathbf{n}=1$ one has $\lambda=2\epsilon(1-\epsilon)/d$. Thus, the temperature decays according to $T(t)=T_0\,(1+t/t_0)^{-2}$, with the initial temperature T_0 and the characteristic time scale $t_0=d/\big[\epsilon(1-\epsilon)\sqrt{T_0}\,\big]$. The temperature quantifies velocity fluctuations. In the following, we focus on the natural case of isotropic velocity distributions. We will show that asymptotically, the temperature represents the only relevant velocity scale as the velocity distribution approaches the scaling form

$$P(\mathbf{v},t)\sim\frac{1}{T^{d/2}}\mathcal{P}\left(\frac{v}{\sqrt{T}}\right). \tag{3}$$

Given the convolution structure of the Boltzmann equation (2), we introduce $F(\mathbf{k},t)$, the Fourier transform of the velocity distribution function, $F(\mathbf{k},t)=\int d\mathbf{v}\, e^{i\mathbf{k}\cdot\mathbf{v}}\, P(\mathbf{v},t)$. Applying the Fourier transform to Eq. (2) and integrating over the velocities gives

$$\frac{1}{\sqrt{T}}\frac{\partial}{\partial t}F(\mathbf{k},t)+F(\mathbf{k},t)=\int d\mathbf{n}\, F\left[\mathbf{k}-\mathbf{q},t\right]F\left[\mathbf{q},t\right], \quad (4)$$

with $\mathbf{q}=(1-\epsilon)(\mathbf{k}\cdot\mathbf{n})\,\mathbf{n}$. This rate equation reflects the momentum transferred between particles during collisions. We seek an isotropic scaling solution for the Fourier transform, the equivalent of (3),

$$F(\mathbf{k},t)=\Phi\left(k^2T\right), \quad (5)$$

with $k\equiv|\mathbf{k}|$. In the $k\to 0$ limit, the Fourier transform, $F(\mathbf{k},t)\cong 1-\frac{1}{2}\,k^2\,T$, implies that the first two terms in the Taylor expansion of the corresponding scaling function $\Phi(x)$ are universal, $\Phi(x)\cong 1-\frac{1}{2}x$.

Let us first consider the simpler 1D case. Equation (4) reduces to $\frac{1}{\sqrt{T}}\frac{\partial}{\partial t}F(k,t)+F(k,t)=F[\epsilon k,t]\,F[k-\epsilon k,t]$ and the scaling function (5) satisfies $-\lambda\, x\,\Phi'(x)+\Phi(x)=\Phi\left[\epsilon^2 x\right]\,\Phi\left[(1-\epsilon)^2x\right]$ [17]. This equation admits a very simple solution [18]

$$\Phi(x)=\left[1+\sqrt{x}\,\right]\,e^{-\sqrt{x}}. \quad (6)$$

The inverse Fourier transform gives the scaling function of the velocity distribution as a squared Lorentzian

$$\mathcal{P}(w)=\frac{2}{\pi}\frac{1}{(1+w^2)^2}. \quad (7)$$

Therefore, the form of the scaling solution (7) is universal as it is independent of the dissipation degree. Another important feature is the algebraic tail of the velocity distribution, $\mathcal{P}(w)\sim w^{-4}$ as $w\to\infty$.

We now return to the general d-dimensional case. Substituting (5) into (4) and using the temperature cooling equation $\frac{d}{dt}T=-\lambda T^{3/2}$ one finds that the scaling function $\Phi(x)$ satisfies $\Phi(x)-\lambda x\Phi'(x)=\int d\mathbf{n}\,\Phi(\xi x)\Phi(\eta x)$, where $\xi=1-(1-\epsilon^2)n_1^2$ and $\eta=(1-\epsilon)^2n_1^2$. To integrate over the impact direction $\mathbf{n}$ we use spherical coordinates and treat the first axis as the polar axis, $n_1=\cos\theta$. The θ-dependent factor of the measure is $d\mathbf{n}=N^{-1}(\sin\theta)^{d-2}d\theta$ with the factor $N=\int_0^\pi(\sin\theta)^{d-2}d\theta=B\left(\frac{1}{2},\frac{d-1}{2}\right)$ ensuring proper normalization ($B(a,b)$ is the beta function). Using $\mu=\cos^2\theta$ as the integration variable, the above governing equation for the scaling function reads

$$-\lambda\, x\,\Phi'(x)+\Phi(x)=\int_0^1\mathcal{D}\mu\;\Phi(\xi x)\;\Phi(\eta x), \quad (8)$$

where $\xi\equiv\xi(\epsilon,\mu)=1-(1-\epsilon^2)\mu$ and $\eta\equiv\eta(\epsilon,\mu)=(1-\epsilon)^2\mu$. Additionally, the integration measure was rewritten using $\mathcal{D}\mu$, defined via $B\left(\frac{1}{2},\frac{d-1}{2}\right)\mathcal{D}\mu=\mu^{-\frac{1}{2}}\,(1-\mu)^{\frac{d-3}{2}}\,d\mu$ (it remains normalized, $\int_0^1\mathcal{D}\mu=1$). In the elastic case, the velocity distribution is Maxwellian. Indeed, $\xi+\eta=1$ and $\lambda=0$ when $\epsilon=0$, and thence $\Phi(x)=e^{-x/2}$.

Our primarily goal is to determine statistics of extremely fast particles, namely the tail of the velocity distribution. This can be accomplished by noting that the large-v behavior of the velocity distribution is reflected by the small-k behavior of its Fourier transform. For example, the small-x expansion of the 1D solution (6) contains both regular and singular terms: $\Phi(x)=1-\frac{1}{2}\,x+\frac{1}{3}\,x^{3/2}+\cdots$, and the dominant singular $x^{3/2}$ term reflects the w^{-4} tail of $\mathcal{P}(w)$. In general, an algebraic tail of the velocity distribution (3),

$$\mathcal{P}(w)\sim w^{-\sigma}\qquad\text{when}\qquad w\to\infty, \quad (9)$$

indicates the existence of a singular component in the Fourier transform,

$$\Phi_{\rm sing}(x)\sim x^{(\sigma-d)/2}\qquad\text{when}\qquad x\to 0, \quad (10)$$

and vice versa. This is seen by recasting the Fourier transform $\Phi(x)\propto\int_0^\infty dw\,w^{d-1}\mathcal{P}(w)\,e^{iw\sqrt{x}}$, into the Laplace transform $I(s)\propto\int_0^\infty dw\,w^{d-1}\mathcal{P}(w)\,e^{-ws}$ using $x=-s^2$. The small-s expansion of the integral $I(s)$ contains regular and singular components. For example, when $\sigma<d$, the integral $I(s)$ diverges as $s\to 0$ and integration over large-w yields the dominant contribution $I_{\rm sing}(s)\sim s^{\sigma-d}$. When $d<\sigma<d+1$, $I(0)$ is finite, but the next term is the above singular term, so $I(s)=I(0)+I_{\rm sing}(s)+\cdots$. In general, the singular contribution is $I_{\rm sing}(s)\sim s^{\sigma-d}$, thereby leading to the singular term of Eq. (10).

The exponent σ can now be obtained by inserting $\Phi(x)=\Phi_{\rm reg}(x)+\Phi_{\rm sing}(x)$ into Eq. (8) and equating the dominant singular terms. Combining $\Phi_{\rm reg}(0)=1$, with the anticipated leading singular term of Eq. (10), we find that the exponent σ is a root of the following integral equation

$$1-\lambda\,\frac{\sigma-d}{2}=\int_0^1\mathcal{D}\mu\;\left[\xi^{(\sigma-d)/2}+\eta^{(\sigma-d)/2}\right]. \quad (11)$$

This equation has a trivial solution $\sigma=d+2$, following from the identity $\int\mathcal{D}\mu\,(\xi+\eta)=1-\lambda$, where the singular and the regular components simply coincide, $x^{(\sigma-d)/2}=x$. Since we seek the leading *singular* term, the solution of Eq. (11) must therefore satisfy $\sigma>d+2$.

The integral equation (11) can be written explicitly in terms of special functions

$$1-\epsilon(1-\epsilon)\,\frac{\sigma-d}{d}= \quad (12)$$
$${}_2F_1\left[\frac{d-\sigma}{2},\frac{1}{2};\frac{d}{2};1-\epsilon^2\right]+(1-\epsilon)^{\sigma-d}\,\frac{\Gamma\left(\frac{\sigma-d+1}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{\sigma}{2}\right)\,\Gamma\left(\frac{1}{2}\right)},$$

with ${}_2F_1(a,b;c;z)$ the hypergeometric function [21]. Interestingly, the exponent $\sigma\equiv\sigma(d,\epsilon)$ is a root of the transcendental equation (12) and thence it depends in a nontrivial fashion on spatial dimension d and the dissipation parameter ϵ.

We first consider the dependence on the dissipation parameter by considering the quasi-elastic limit $\epsilon \to 0$. As discussed above, in the elastic case the velocity distribution is Maxwellian and the Fourier transform is simply $\Phi(x) = e^{-x/2}$ [22]. This implies a diverging exponent $\sigma \to \infty$ as $\epsilon \to 0$. Therefore, the right-hand side of Eq. (12) vanishes, and the leading behavior is

$$\sigma \simeq \frac{d}{\epsilon} \qquad \text{as} \qquad \epsilon \to 0. \tag{13}$$

One can further expand $\sigma(d,\epsilon)$ in the $\epsilon \to 0$ limit to find: $\sigma(d,\epsilon) = d\,\epsilon^{-1} + a_1(d)\,\epsilon^{-1/2} + a_2(d) + \cdots$. We merely quote the leading correction in the physically relevant spatial dimensions $a_1(2) = -2(e^{-2}+1)/\sqrt{\pi}$ and $a_1(3) = -\sqrt{3\pi/2}$. Clearly, the quasi-elastic limit is singular. Dissipation, even if minute, seriously changes the nature of the system [6,23].

Next, we discuss the dependence on the dimension. First, one can verify that $\sigma = 4$ when $d = 1$ using the identity ${}_2F_1(a,b;b;z) = (1-z)^{-a}$. The case $d = 1$ is unique in that the entire scaling function and in particular the exponent do not dependent on ϵ. The case of $d \to \infty$ is similar to the $\epsilon = 0$ case in that the inelastic nature of the collisions becomes irrelevant, the velocity distribution is Maxwellian, and the exponent diverges, $\sigma \to \infty$ as $d \to \infty$. In this limit, the second integral in Eq. (11) is negligible as it vanishes exponentially with the dimension. The first integral can be evaluated by taking the limits $d \to \infty$ and $\mu \to 0$, with $z = \mu d/2$ being finite. Then, the integration measure is transformed $\mathcal{D}\mu \to (\pi z)^{-1/2} e^{-z}\,dz$, and Eq. (11) becomes $1 - \epsilon(1-\epsilon)u = \int_0^\infty dz (\pi z)^{-1/2} e^{-[1+(1-\epsilon^2)u]z}$ where $u = \frac{\sigma}{d} - 1$. Performing the integration yields $1 - \epsilon(1-\epsilon)u = [1 + (1-\epsilon^2)u]^{-1/2}$, from which we find u and

$$\sigma \simeq d\,\frac{1 + \frac{3}{2}\epsilon - \epsilon^3 - \epsilon^{1/2}\left(1 + \frac{5}{4}\epsilon\right)^{1/2}}{\epsilon(1-\epsilon^2)}, \tag{14}$$

as $d \to \infty$. In general, $\sigma \propto d$, and therefore, the algebraic decay becomes sharper as the dimension increases. The exponent $\sigma(d,\epsilon)$ increases monotonically with increasing dimension, and additionally, it increases monotonically with decreasing ϵ (see Fig. 1). Both features are intuitive as they mirror the monotonic dependence of the energy dissipation rate $\lambda = 2\epsilon(1-\epsilon)/d$ on d and ϵ. Hence, the completely inelastic case provides a lower bound for the exponent, $\sigma(d,\epsilon) \geq \sigma(d, \epsilon = 1/2)$ with $\sigma(d, 1/2) = 6.28753,\ 8.32937$, for $d = 2,\ 3$, respectively. The former value should be compared with $\sigma \approx 5$, obtained from numerical simulations [18]. The algebraic tails are characterized by unusually large exponents which may be difficult to measure accurately in practice (for typical granular particles $\epsilon \approx 0.1$ yielding $\sigma \approx 30$). Figure 1 also shows that the quantity σ/d weakly depends upon the dimension, and the large-d limit (14) provides a useful approximation.

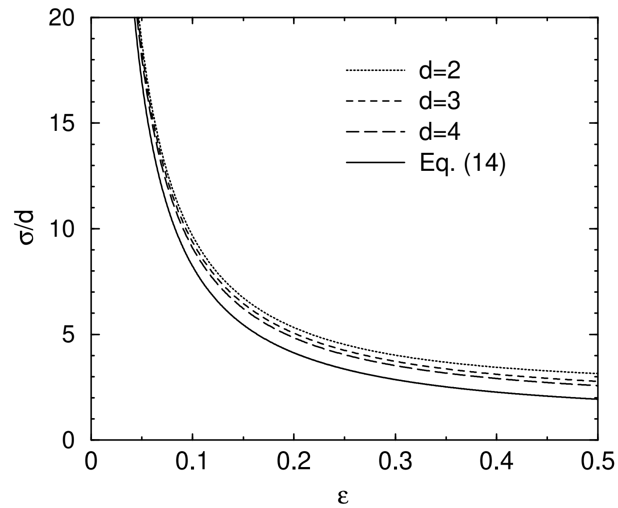

FIG. 1. The exact exponent σ, obtained from Eq. (11), versus the dissipation parameter ϵ. The exponent was scaled by the dimension d. Shown also is the limiting large dimension expression (14).

Thus far, we discussed only freely cooling systems where the energy decreases indefinitely. In typical experimental situations, however, the system is supplied with energy to balance the energy dissipation [10,11]. Theoretically, it is natural to consider white noise forcing [9], i.e., coupling to a thermal heat bath which leads to a nonequilibrium steady state. Specifically, we assume that in addition to changes due to collisions, velocities may also change due to an external forcing: $\frac{dv_j}{dt}|_{\rm heat} = \xi_j$ with $j = 1, \ldots, d$. We use standard uncorrelated white noise $\langle \xi_i(t)\xi_j(t')\rangle = 2D\delta_{ij}\delta(t-t')$ with a zero average $\langle \xi_j \rangle = 0$. The temperature rate equation is modified by the additional source term $\frac{d}{dt}T + \lambda T^{3/2} = 2D$, and the system approaches a steady state, $T_\infty = (2D/\lambda)^{2/3}$. The relaxation toward this state is exponential, $|T_\infty - T| \sim e^{-t/\tau}$.

Uncorrelated white noise forcing amounts to diffusion in velocity space, and Eq. (4) is modified as follows $\frac{1}{\sqrt{T}}\frac{\partial}{\partial t} \to \frac{1}{\sqrt{T}}\frac{\partial}{\partial t} + Dk^2$. In the steady state, the Fourier transform, $F(\mathbf{k}, t = \infty) \equiv \Psi(y)$ with $y = Dk^2$, obeys

$$(1+y)\,\Psi(y) = \langle \Psi(\xi y)\,\Psi(\eta y)\rangle, \tag{15}$$

where integration with respect to the measure $\mathcal{D}\mu$ is denoted by $\langle f \rangle = \int_0^1 \mathcal{D}\mu f(\mu)$.

Equation (15) is solved recursively by employing the cumulant expansion $\Psi(y) = \exp\left[\sum_{n\geq 1}(-y)^n F_n\right]$. Writing $1 + y = \exp\left[\sum_{n\geq 1}(-y)^n/n\right]$, we recast Eq. (15) into

$$1 = \left\langle \exp\left[\sum_{n=1}^{\infty}(-y)^n(n^{-1} - G_n)\right]\right\rangle, \tag{16}$$

where $G_n = F_n(1-\xi^n-\eta^n)$. The desired cumulants F_n are obtained by solving for $\langle G_n\rangle$ recursively and then using $F_n = \langle G_n\rangle/\langle 1-\xi^n-\eta^n\rangle$. In one dimension $\langle\mu^n\rangle = 1$ and one immediately obtains $\langle G_n\rangle = n^{-1}$ [17]. In higher dimensions, the averages acquire non-trivial dependence on n [24] though the qualitative nature of the solution remains the same since $\langle G_n\rangle$ and hence the cumulants F_n (as well as the moments of the distribution) are positive. This implies the following small-k behavior of the Fourier transform at the steady state: $\ln F_\infty(\mathbf{k}) \sim -(k/k_0)^2$. This behavior agrees with the prediction of Ref. [25] derived via a small-ϵ expansion, but differs from the stretched exponential behavior $\exp(-v^{3/2})$ found for the driven inelastic hard sphere gas [9].

In summary, we have studied inelastic gases within the framework of the Boltzmann equation with a uniform collision kernel. In the freely evolving case, we have shown analytically that the density of high-energy particles is suppressed algebraically. The algebraic tails are characterized by remarkably large exponents, and may be hard to distinguish from (stretched) exponential tails. Our results, combined with previous kinetic theory studies which find exponential, stretched exponential, and Gaussian tails, indicate that the extremal characteristics can be very sensitive to parameters such as the restitution coefficient, and the dimension [8,9,13]. On the other hand, our results in the forced case support the near-Maxwellian assumptions typically used to obtain macroscopic transport coefficients from kinetic theory [14].

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