Nucleation-and-Growth in One Dimension

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We study statistical properties of the Kolmogorov-Avrami-Johnson-Mehl nucleation-and-growth model in one dimension. We obtain exact results for the gap density as well as the island distribution. When all nucleation events occur simultaneously, we show that the island distribution has discontinuous derivatives on the rays $x_n(t) = nt$, $n = 1, 2, 3 \ldots$ We introduce an accelerated growth mechanism with growth rate increasing linearly with the island size. We solve for the inter-island gap density and show that the system reaches complete coverage in a finite time and that the near-critical behavior of the system is robust, *i.e.*, it is insensitive to details such as the nucleation mechanism.

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I. INTRODUCTION

Inhomogeneous systems where stable and metastable regions coexist are common in nature. Phase separation and coarsening [1], aggregation [2], wetting [3], dendritic growth [4], and growth of breath figures [5] are just a few examples of such systems. Typically, the stable phase grows into the metastable phase according to complicated kinematic rules. However, in certain cases such as adsorption [6], simple ballistic growth rules apply. The Kolmogorov-Avrami-Johnson-Mehl (KAJM) nucleation-and-growth process is a natural model incorporating nucleation of stable phases with ballistic growth [6-17].

In this work we present exact results for various statistical properties of the KAJM growth model in one dimension. The process depends on the nucleation rate as well as the initial concentration of the growing phase. There are two limiting cases, instantaneous nucleation (subsequent nucleation rate vanishes) and continuous nucleation (vanishing initial concentration). The growth rate of the stable phase may also depend on size of the growing islands. Besides the ordinary KAJM model with size-independent growth rate, the accelerated KAJM model with growth rate linear in size is also solvable as we shall demonstrate in this paper.

To solve the KAJM model with linear growth rate, a deeper understanding of the island-size distribution is necessary. Surprisingly, even for the classical KAJM model little is known about the island-size distribution function. Hence, we we first investigate the KAJM model with a constant growth velocity. We introduce the density of islands containing n "seeds" and show that this distribution is not a smooth function of the space variable. As a result, the total island length distribution has spatial discontinuous derivatives at every integer multiple of t. In the continuous nucleation case, we obtain only the inverse Laplace transform of this generalized island distribution. However, an asymptotic analysis shows that the relative fraction of islands containing n seeds decays algebraically in time rather than exponentially as in the case of instantaneous nucleation.

In the second part of our study, we introduce an accelerated nucleation-and-growth process where the growth velocity depends on the island size. This growth mechanism is motivated by an accelerated random sequential adsorption (RSA) process [18]. In the ordinary RSA of monomers on a lattice an adsorption attempt on a site is successful only if that site is empty. Unlike ordinary RSA where an attempt to adsorb on an occupied site is rejected, in the accelerated process any attempt is successful – if a monomer is deposited onto an already existing island it diffuses until it reaches an empty site on the island boundary. Hence, islands grow with a rate increasing with island size; if the diffusion time scale is very small compare to the adsorption time scale, the growth rate becomes a linear function of the island length. The continuum version of this model is simply the KAJM nucleation-and-growth process with growth rate linear in the island size. While for the lattice model only an approximate theory exists, we generalize the KAJM theory to the accelerated growth model. Exact results for the island gap distribution show that the system is covered in a finite time. Also, the behavior near complete coverage is robust. It is independent of many details of the growth velocity as well as the nucleation mechanism; it is the same for instantaneous and continuous nucleation.

The rest of this paper is organized as follows. In Section II, we consider the ordinary KAJM nucleation-and-growth process. We first review the existing theory, and present a summary of the exact results for both instantaneous and continuous nucleation. We then consider the detailed island gap density and analyze its properties. In Section III, we introduce the accelerated growth model and solve for the exact inter-island gap distribution. Additionally, we analyze the behavior close to complete coverage.

II. THEORY OF NUCLEATION-AND-GROWTH IN ONE DIMENSION

In the ordinary Kolmogorov-Avrami-Johnson-Mehl model size-less islands ("seeds") nucleate randomly in space with rate $\gamma(t)$ per unit length, and grow with constant velocity, which we set equal to $1/2$, in both positive and negative direction. A collision between two such growing islands results in a similarly growing island whose length is given by the sum of its constituents' lengths. Since islands grow with unit rate, the system is covered with a rate proportional to the density of islands, $N(t)$; in other words, the fraction of uncovered space, $S(t)$, satisfies the rate equation

$$\frac{dS(t)}{dt} = -N(t). \tag{1}$$

Let us introduce $f(x,t)$, the density of inter-island gaps of size x at time t. The total island density and the uncovered fraction are simply given by $N(t) = \int_0^\infty dx\, f(x,t)$, and $S(t) = \int_0^\infty dx\, x f(x,t)$, respectively. The gap distribution evolves according to the following rate equation

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial f(x,t)}{\partial x} - \gamma(t) x f(x,t) + 2\gamma(t)\int_x^\infty dy\, f(y,t). \tag{2}$$

The first term in the right-hand side of Eq. (2) accounts for the shrinking of a gap caused by growth of its two neighboring islands. The last two terms represent loss (gain) of gaps due to nucleation of seeds. The rate of change in the island density is evaluated by integrating equation (2)

$$\frac{dN(t)}{dt} = -f(0,t) + \gamma(t)S(t). \tag{3}$$

Additionally, multiplying Eq. (2) by x and integrating, we recover Eq. (1) thus providing a check of self-consistency.

A complementary distribution is $g(x,t)$, the density of islands of size x at time t. The number density and the fraction of the uncovered space can be alternatively expressed via the island distribution: $N(t) = \int_0^\infty dx\, g(x,t)$ and $S(t) = 1 - \int_0^\infty dx\, x g(x,t)$, respectively. The density $g(x,t)$ obeys [16]

$$\frac{\partial g(x,t)}{\partial t} = -\frac{\partial g(x,t)}{\partial x} + \gamma(t)S(t)\delta(x) + a(t)\left[\int_0^x dy\, g(y,t)g(x-y,t) - 2N(t)g(x,t)\right]. \tag{4}$$

While the first term in the right-hand side of Eq. (4) corresponds to growth with unit velocity, the second term represents creation of size-less islands due to nucleation. The last two terms describe coalescence events between two growing islands. Integrating equation (4), one finds that $N(t) = \int_0^\infty dx\, g(x,t)$ satisfies the rate equation $dN(t)/dt = -a(t)N(t)^2 + \gamma(t)S(t)$. Comparing this expression with Eq. (3), the prefactor $a(t)$ is found, $a(t) = f(0,t)/N(t)^2$.

Another interesting quantity is $g_n(x,t)$, the density of islands of length x that contain n seeds. The previous island size distribution is obtained from this more detailed quantity by a simple summation $g(x,t) = \sum_{n\geq 1} g_n(x,t)$. The $g_n(x,t)$ distribution obeys a generalization to Eq. (4):

$$\frac{\partial g_n(x,t)}{\partial t} = -\frac{\partial g_n(x,t)}{\partial x} + \gamma(t)S(t)\delta_{n,1}\delta(x) \;+ a(t)\left[\sum_{m=1}^{n-1}\int_0^x dy\, g_m(y,t)g_{n-m}(x-y,t) - 2N(t)g_n(x,t)\right]. \tag{5}$$

This equation simply reflects the fact that the number of seeds is conserved during collisions.

To solve the above equation, it is useful to introduce the Laplace transform of the generating functions of $g_n(x,t)$ defined as $g(s,z,t) = \int_0^\infty dx\, \sum_{n\geq 1} z^n e^{-sx} g_n(x,t)$. This "joint transform" satisfies

$$\frac{\partial g(s,z,t)}{\partial t} = -[s + 2a(t)N(t)]g(s,z,t) + z\gamma(t)S(t) + a(t)g(s,z,t)^2 \tag{6}$$

Much information concerning the island distribution can be directly extracted from $g(s,z,t)$. For example, the Laplace transform of the island distribution, $g(s,t) = \int_0^\infty dx\, e^{-sx} g(x,t)$, can be readily found from the joint Laplace transform, $g(s,t) \equiv g(s, z=1, t)$. The total density of islands containing n seeds, $g_n(t) = \int_0^\infty dx\, g_n(x,t)$, is obtained by considering a vanishing s, namely, $g(z,t) = \sum_{n\geq 1} z^n g_n(t) = g(s=0,z,t)$.

Two natural limits of KAJM nucleation-and-growth process are instantaneous nucleation and continuous nucleation. In instantaneous nucleation all seeds start growing at the same time, taken as $t=0$ for convenience. In continuous nucleation the space contains no seeds initially and seeds appear uniformly in space and time on yet uncovered space. While for the former case, once the positions of the seeds are specified the growth is fully deterministic, in the latter case the process is stochastic. We now present exact results for both cases using the above formalism.

A. Instantaneous Nucleation

In instantaneous nucleation, $\gamma(t)=\Gamma\delta(t)$ since all nucleation events occur simultaneously at $t=0$. Hence for $t>0$, Eq. (2) can be rewritten as $\partial f(x,t)/\partial x_-=0$ with $x_\pm\equiv x\pm t$, and the gap distribution is thus a function of the variable $x_+=x+t$ only. Let the initial gap distribution be $f_0(x)=f(x,t=0)$, then the solution for an arbitrary initial distribution is readily found $f(x,t)=f_0(x+t)$. We set $\Gamma=1$ without loss of generality, and furthermore, we restrict our attention to the initial conditions where the seeds are distributed uniformly in space, $f_0(x)=e^{-x}$. In this case, the solution $f(x,t)=f_0(x+t)$ becomes

$$f(x,t)=e^{-x-t}. \tag{7}$$

The island number density as well as the uncovered fraction decay exponentially in time $N(t)=S(t)=e^{-t}$. The average island length can be easily found as well $\langle x(t)\rangle=\big(1-S(t)\big)/N(t)=e^t-1$. These average quantities were originally derived from simple considerations [7,8]. For example, the uncovered fraction equals the probability that a point, say the origin, remains uncovered at time t. For such an event to occur, the interval $[-t/2,t/2]$ must contain no seeds initially, and hence the e^{-t} decay. Combining $S(t)=e^{-t}$ with Eq. (1) yields the number density $N(t)=e^{-t}$. These considerations are applicable in arbitrary dimensions while more complete analytical results for the gap and the island distributions are limited to one dimension.

The Laplace transform of the joint island distribution, $g(s,z,t)$, satisfies

$$\frac{\partial g(s,z,t)}{\partial t}=-(s+2)g(s,z,t)+e^t g(s,z,t)^2. \tag{8}$$

The above was obtained from Eq. (6) by substituting the appropriate prefactor $a(t)=f(0,t)/N(t)^2=e^t$. Solving Eq. (8) subject to the initial conditions $g(s,z,t=0)=z$ gives

$$g(s,z,t)=\frac{ze^{-(s+2)t}}{1-z[(1-e^{-(s+1)t})/(s+1)]}. \tag{9}$$

The Laplace transform of the island distribution is readily found by evaluating $g(s,z,t)$ at $z=1$,

$$g(s,t)=e^{-t}\frac{s+1}{se^{(s+1)t}+1}. \tag{10}$$

One can immediately recover the total number density, $N(t)=g(s=0,t)=e^{-t}$, and the fraction of uncovered space, $S(t)=1-\partial g(s,t)/\partial s|_{s=0}=e^{-t}$. The inverse Laplace transform can be obtained by expanding the Laplace transform in powers of e^{-st}, *i.e.*, $g(s,t)=\sum_{m\ge1}\tilde g_m(s,t)e^{-mst}$. Performing the inverse Laplace transform term by term yields

$$g(x,t)=e^{-2t}\left[\delta(x-t)+\theta(x-t)\right]+\sum_{n=1}^{\infty}(-1)^n e^{-(n+2)t}\theta\left(x-(n+1)t\right)\left[\frac{(x-(n+1)t)^{n-1}}{(n-1)!}+\frac{(x-(n+1)t)^n}{n!}\right], \tag{11}$$

where $\theta(x)$ is the Heavyside step function. Unlike the gap distribution which is a simple smooth function, $f(x,t)=e^{-x-t}$, the island distribution is more complex. Although it is a continuous function, it has discontinuous derivatives on the rays $x_n(t)=nt$, for $n=1,2,3\ldots$ These discontinuities are exponentially suppressed because of the e^{-nt} term and should be noticeable only for small n in the long time limit. This situation is reminiscent of the behavior of extremal properties of stochastic systems such as random walks and fragmentation models [19]. The tail of the distribution can be found directly from the Laplace transform. Taking the $s=0$ limit of equation (11) and performing the inverse Laplace transform yields

$$g(x,t)\sim\langle x\rangle^{-2}e^{-x/\langle x\rangle},\qquad x\to\infty, \tag{12}$$

with $\langle x\rangle\sim e^t$. Hence, the tail of the island length distribution approaches an exponential distribution with an exponentially growing average. The term $e^{-2t}\delta(x-t)$ in equation (11) arises from islands that contain a single seed. Thus, the total number density of such islands is $g_1(t)=e^{-2t}$, or equivalently, the fraction of one-seed islands decays exponentially in time $p_1(t)\equiv g_1(t)/N(t)=e^{-t}$. The rest of the $g_n(t)$ distribution is easily obtained from the joint generating function by evaluating the $s=0$ limit, $\sum_{n\ge1}g_nz^n=g(s=0,z,t)$ and expanding in powers of z^n. Thus we find for the fraction of n-seeds islands

$$p_n(t)\equiv\frac{g_n(t)}{N(t)}=e^{-t}(1-e^{-t})^{n-1}. \tag{13}$$

The average number of seeds, $\langle n(t)\rangle = \sum_{n\geq 1} np_n(t)$, is readily found to be $\langle n(t)\rangle = e^t$. Therefore in the scaling limit, $t\to\infty$ with $n/\langle n(t)\rangle$ kept finite, the exact distribution of Eq. (13) becomes

$$p_n(t) \simeq \langle n(t)\rangle^{-1} e^{-n/\langle n(t)\rangle}. \tag{14}$$

To find the complete $g_n(x,t)$ distribution one first determines the Laplace transform $g_n(s,t) = \int_0^\infty dx\, e^{-sx} g_n(x,t)$ by expanding $g(s,z,t)$ in powers of z^n,

$$g_n(s,t) = e^{-(s+2)t}\left(\frac{1-e^{-(s+1)t}}{s+1}\right)^{n-1}. \tag{15}$$

Then one has to perform the inverse Laplace transform. Simple explicit expressions are found for small n, *e.g.*, the distribution of "monomers" is indeed $g_1(x,t) = e^{-2t}\delta(x-t)$ as we have already seen previously, and the distribution of dimers is $g_2(x,t) = e^{-(x+t)}[\theta(x-t)-\theta(x-2t)]$. Generally for $n\geq 2$, one finds

$$g_n(x,t) = (n-1)e^{-(x+t)}\sum_{m=0}^{n-1}\frac{(-1)^m[x-(m+1)t]^{n-2}\theta[x-(m+1)t]}{m!(n-1-m)!}. \qquad t\leq x\leq nt \tag{16}$$

The distribution $g_n(x,t)$ vanishes outside the interval $[t,nt]$. An island of maximal length nt results from an initial arrangement of the n seeds where all nearest neighbor distances equal t. Similar to the island distribution, $g_n(x,t)$ has n discontinuous derivatives at $x=t,2t,\ldots,nt$.

It is also useful to consider the average length of an island containing n seeds, $\langle x\rangle_n = -\partial \ln g_n(s,t)/\partial s|_{s=0} = t+(n-1)[1-t/(e^t-1)]$. As a result, the leading behavior in both limiting cases is linear in time

$$\langle x_n(t)\rangle \simeq \begin{cases} (n+1)t/2, & t\to 0;\\ (n-1)+t, & t\to\infty.\end{cases} \tag{17}$$

The prefactor depends on n initially, and at the later stages of the growth process it becomes independent of n. The width of the distribution $\sigma_n^2 = \langle x^2\rangle - \langle x\rangle^2$ is conveniently obtained from Eq. (15), $\sigma_n^2(t) = \partial^2 \ln g_n(s,t)/\partial s^2|_{s=0}$, and we quote only the more interesting long time asymptotics. In this limit, the width of the distribution approaches a constant, $\sigma_n^2(t)\to n-1$ as $t\to\infty$.

B. Continuous Nucleation

We now turn to the case of continuous nucleation, $\gamma(t) = \text{const}$, and set $\gamma=1$ for notational simplicity. As a preliminary step, we solve for the gap distribution $f(x,t)$ subject to the initial conditions

$$\int_0^\infty dx\, f(x,0) = 0, \qquad \int_0^\infty dx\, x f(x,0) = 1, \tag{18}$$

corresponding to no seeds present at $t=0$. Substituting the ansatz $f(x,t) = \phi(t)e^{-xt}$ eliminates the size dependence from the rate equation (2). The time dependent prefactor $\phi(t)$ satisfies the following ordinary differential equation, $d\phi(t)/dt = \phi(t)(2/t-t)$. Solving this subject to the initial conditions of Eq. (18) yields $\phi(t) = t^2 e^{-t^2/2}$, and thus the gap distribution is given by

$$f(x,t) = t^2 e^{-xt-t^2/2}. \tag{19}$$

Integrating over the space variable gives the island number density, the uncovered fraction, and the average island length:

$$N(t) = te^{-t^2/2}, \qquad S(t) = e^{-t^2/2}, \qquad \langle x(t)\rangle = \frac{e^{t^2/2}-1}{t}. \tag{20}$$

These average quantities can be found from simple considerations as well. The uncovered fraction equals the probability that a point, say the origin, remains uncovered during the time interval $[0,t]$. For this event to happen, no nucleation events can occur at a point x, $0<|x|<t/2$ during the time interval $[0,2|x|]$. The probability for such an event is indeed $S(t) = e^{-t^2/2}$. The number density can also be easily found using the relation $dS(t)/dt = -N(t)$.

Once the prefactor $a(t) = f(0,t)/N(t)^2 = e^{t^2/2}$ is known the equation satisfied by the joint transform of the gap distribution $g(s,z,t)$ can be written

$$\frac{\partial g(s,z,t)}{\partial t} = -[s+2t]g(s,z,t) + ze^{-t^2/2} + e^{t^2/2}g(s,z,t)^2. \tag{21}$$

The transformation

$$g(s,z,t) = e^{-t^2/2}\left[\frac{\eta}{2} - \frac{\psi'(\eta)}{\psi(\eta)}\right], \qquad \eta = s+t, \tag{22}$$

(prime denotes the derivative with respect to η) reduces Eq. (21) to the parabolic cylinder equation [20] for the auxiliary function $\psi(\eta)$, namely $\psi'' + (z - 1/2 - \eta^2/4)\psi = 0$. Finally, the generating function is found

$$g(s,z,t) = e^{-t^2/2}\left[\frac{t+s}{2} - \frac{D'_{z-1}(s+t) + D'_{z-1}(-s-t)}{D_{z-1}(s+t) + D_{z-1}(-s-t)}\right] \tag{23}$$

with D_{z-1} the parabolic cylinder function of order $z-1$. It can be verified that $N(t) = g(0,1,t) = te^{-t^2/2}$. Two important cases are the Laplace transform of the length distribution function [16]

$$g(s,t) = g(s,z=1,t) = e^{-t^2/2}\left[s + t - \frac{se^{t^2/2+st}}{1 + s\int_0^t d\tau e^{\tau^2/2+s\tau}}\right], \tag{24}$$

and the generating function $g(z,t) = \sum_{n\geq 1} z^n g_n(t)$

$$g(z,t) = g(s=0,z,t) = e^{-t^2/2}\left[\frac{t}{2} - \frac{D'_{z-1}(t) + D'_{z-1}(-t)}{D_{z-1}(t) + D_{z-1}(-t)}\right]. \tag{25}$$

We are unable to perform the inverse Laplace transform. However, by a direct solution of equation (5), it is still possible to obtain several quantities. For example, the one-seed island distribution obeys

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) g_1(x,t) = -2tg_1(x,t) + e^{-t^2/2}\delta(x). \tag{26}$$

Solving this partial differential equation gives

$$g_1(x,t) = e^{-t^2}e^{(x-t)^2/2} \qquad 0 < x < t, \tag{27}$$

and $g(x,t) = 0$ outside the space interval $[0,t]$. The above distribution is strongly peaked ate $x = t$. A numerical solution of Eq. (4) shows that the total gap distribution, $g(x,t)$, is also sharply peaked at this point [16]. In fact, we expect that one-seed islands dominate for sizes approaching t from below. Following the previous section findings, we also expect that the n-seed densities are discontinuous on the rays $x_n(t) = nt$. However, such discontinuous behavior might be hardly visible for large n.

The density of one-seed island equals $g_1(t) = e^{-t^2}\int_0^t d\tau\, e^{\tau^2/2}$. Evaluating the limit $t \to \infty$ gives $g_1(t) \simeq t^{-1}e^{-t^2/2}$. Unlike the simultaneous case, where the fraction of one-seed islands decayed exponentially, here we find that the quantity $p_1(t) \equiv g_1(t)/N(t) \simeq t^{-2}$ decays significantly slower in an algebraic fashion. This power-law behavior is actually a general one. Writing $g_n(t) = e^{-t^2/2}\tilde{g}_n(t)$ and integrating equation (5) with respect to x, we find

$$\frac{d\tilde{g}_n(t)}{dt} = \delta_{n,1} - t\tilde{g}_n(t) + \sum_{m=1}^{n-1}\tilde{g}_m(t)\tilde{g}_{n-m}(t). \tag{28}$$

Instead of solving the above equation generally, we obtain the leading asymptotic behavior in the limits of small and large t. In the long time limit, the left hand side is negligible and can be safely discarded. In the limit of small time, the loss term $-t\tilde{g}_n(t)$ is unimportant. Solving the resulting approximate equations gives

$$p_n(t) = \frac{g_n(t)}{N(t)} \simeq \begin{cases} a_n t^{2(n-1)}, & t \to 0; \\ b_n t^{-2n}, & t \to \infty; \end{cases} \tag{29}$$

with the prefactors $a_n = 2^{2n}(2^{2n}-1)B_n/(2n)!$ (B_n are the Bernoulli numbers) and $b_n = (2n)!/[2(2n-1)n!^2]$. While the early time behavior resembles the simultaneous case, the long time behavior is algebraic rather than exponential. Note also the nontrivial dependence of both prefactors, a_n and b_n, on the number of seeds n.

III. ACCELERATED NUCLEATION-AND-GROWTH

Most of the studies of the nucleation-and-growth processes assume that the growth velocity is constant (see *e.g.* [16] and references therein). In this Section we demonstrate that a particular generalized KAJM model with *accelerated* growth, namely with velocity linear in the island size, can be treated analytically in one dimension. We emphasize that the linear dependence of the growth velocity on island size naturally appears in several problems. One application which has already been mentioned in the Introduction concerns the accelerated random sequential adsorption (RSA) on a line with precursor layer diffusion. Indeed, in accelerated RSA particles are deposited onto the substrate and they occupy empty sites. Particles that are deposited on occupied sites (extrinsic precursor state) can lose enough kinetic energy, such that they do not desorb back to the gas phase. Instead, they diffuse on top of occupied islands until they encounter an empty site on the island boundary where they are deposited irreversibly [21,18]. If the process is adsorption-limited, *i.e.*, the diffusion time scale is small compare to the adsorption time scale, islands on the 1D substrate grow with rate proportional to the island length. Another possible application is related to biological growth where seeds are the source of the new phase. Therefore, in 1D one-seed islands grow with rate 1, two-seed islands grow with rate 2, *etc.* The definitions of both accelerated RSA and the biological model are appealingly simple, and they exhibit similar behavior to that of the KAJM model with linear growth rate. The latter model has the advantage that it is more amenable to analytical treatment.

Thus, we consider the KAJM model with an accelerated growth mechanism, namely with growth rate proportional to the length of a domain, or better equal to $1+x$ to ensure a finite growth velocity for initial size-less islands. As the length of an isolated domain grows exponentially in time, it is expected that for this accelerated KAJM space is covered in *finite* time t_c. Below, we will find exact values for some interesting quantities, including t_c and the gap distribution. The rate equation for $f(x,t)$ is a generalization of equation (2),

$$\frac{\partial f(x,t)}{\partial t} = [1+\langle x(t)\rangle]\frac{\partial f(x,t)}{\partial x} - \gamma(t)xf(x,t) + 2\gamma(t)\int_x^\infty dy\, f(y,t). \tag{30}$$

The growth term reflects the fact that the growth velocity of islands is linear in their length. The linear dependence is crucial in writing the above equation. A gap between two islands of sizes x_1 and x_2 shrinks with rate $1+(x_1+x_2)/2$, and we can use the equality $\langle x_1+x_2\rangle/2 = \langle x\rangle$. It is possible to write down the rate equation for the island distribution function. However, the analysis of that equation is very cumbersome. Therefore in the following we limit ourselves to the gap distribution which can be examined in depth for both instantaneous and continuous nucleation. We also present an approximate treatment of generalized KAJM models where the growth rate equals an arbitrary power of the island size.

A. Instantaneous Nucleation

We consider first the case of instantaneous nucleation. To solve Eq. (30) with $\gamma(t)=\delta(t)$, it is useful to introduce a modified time variable, $T(t)$, defined by $T(t)=\int_0^t dt'[1+\langle x(t')\rangle]$. In terms of this variable, Eq. (30) simplifies to

$$\left(\frac{\partial}{\partial T} - \frac{\partial}{\partial x}\right) f(x,t) = 0, \tag{31}$$

Similar to the usual KAJM growth, the gap distribution is readily found for arbitrary initial conditions, $f(x,t=0)=f_0(x)$, to yield $f(x,t)=f_0(x+T)$. In particular, let us assume that size-less islands were initially randomly distributed with unit density, $f_0(x)=e^{-x}$. Then the gap distribution reads

$$f(x,t) = e^{-x-T}. \tag{32}$$

Consequently, we get

$$N(T) = S(T) = e^{-T}, \qquad \langle x(T)\rangle = e^T - 1. \tag{33}$$

The average island length was obtained using $\langle x(T)\rangle = \big(1-S(T)\big)/N(T)$. for the number density, the uncovered fraction, and the average island length, respectively. To obtain the explicit time dependence, it is necessary to solve $dT/dt = 1+\langle x(T)\rangle = e^T$, which is integrated to yield $e^{-T}=1-t$. This allows us to determine the time of complete coverage, $t_c=1$. Reexpressing the exact results in terms of the physical time, we arrive at

$$f(x,t) = (1-t)e^{-x}, \qquad N(t) = S(t) = 1-t, \qquad \langle x(t)\rangle = \frac{t}{1-t}. \tag{34}$$

Interestingly, the uncovered fraction and the number density are equal, as for the KAJM model. The covered fraction exhibits a linear growth, $1 - S(t) = t$. This result agrees with the lattice adsorption process of Ref. [18]. Identical behavior is found for the biological growth process where n-seed islands grow with rate n. For simultaneous nucleation, the number of seeds remains constant and hence the covered fraction increases with constant rate. However, the covered fraction is the *only* characteristic which is known analytically for both of these lattice models.

Return now to the accelerated KAJM model and consider the competition between the two components of the velocity: the intrinsic part and the size dependent part. Suppose that size-less islands grow with velocity v_0, *i.e.*, the growth velocity of an x-island equals $v_0 + x$. Then, the rate equation reads

$$\left(\frac{\partial}{\partial t} - [v_0 + \langle x(t)\rangle]\frac{\partial}{\partial x}\right) f(x,t) = 0. \tag{35}$$

The above treatment holds with the modified time variable, $T = \int_0^t [v_0 + \langle x(t')\rangle] dt'$. Following the steps that led to equation (34) gives

$$f(x,t) = \frac{1 - v_0 e^{(1-v_0)t}}{1 - v_0}\, e^{-x}. \tag{36}$$

Additionally, the number density is given by $N(t) = S(t) = (1 - v_0 e^{(1-v_0)t})/(1 - v_0)$, and the average length is $\langle x(t)\rangle = v_0(1 - e^{(1-v_0)t})/(v_0 e^{(1-v_0)t} - 1)$. By taking the limit $v_0 \to 1$, one can verify that the previous results are recovered. The critical time is thus $t_c = \ln v_0/(v_0 - 1)$. Both limiting cases exhibit logarithmic behavior

$$t_c \simeq \begin{cases} \ln(1/v_0), & v_0 \to 0; \\ \ln v_0/v_0, & v_0 \to \infty. \end{cases} \tag{37}$$

The limiting case $v_0 \to 0$ reduces to the KAJM growth where the coverage time is infinite. When $t \to t_c$, the gap distribution vanishes according to

$$f(x,t) \simeq (t_c - t)e^{-x}. \tag{38}$$

It is seen that near-critical behavior such as $S(t) \simeq (t_c - t)$ is independent of the relative magnitudes of the two components of the growth velocity.

B. Continuous Nucleation

We turn now to the continuous nucleation case, $\gamma > 0$. We will assume that the system is initially empty; therefore, the initial conditions are given by Eq. (18). Similar to the previous section, we seek for a solution to Eq. (30) of the form

$$f(x,t) = \phi(t)e^{-\gamma x t}. \tag{39}$$

The choice of the exponential factor allows us to cancel the x-dependent term in the right-hand side of equation (30). The rate equation reduces to an ordinary differential equation,

$$\frac{d\phi(t)}{dt} = 2\phi(t)/t - \gamma\phi(t)t[1 + \langle x(t)\rangle]. \tag{40}$$

Expressing $\langle x(t)\rangle$ via $\phi(t)$ gives $\langle x(t)\rangle = (1 - \int_0^\infty dx\, x f(x,t))/\int_0^\infty dx\, f(x,t) = [1 - \phi(t)/(\gamma t)^2]/[\phi(t)/\gamma t]$. Substituting $\langle x(t)\rangle$ in equation (40) and solving the resulting *closed* differential equation for $\phi(t)$, we obtain

$$f(x,t) = (\gamma t)^2\, e^{-\gamma x t + t - \gamma t^2/2}\left[1 - \int_0^t d\tau\, e^{-\tau + \gamma\tau^2/2}\right]. \tag{41}$$

This exact solution agrees with the initial condition of equation (18).

The instant of complete coverage is determined from the positivity condition of the gap-size distribution function. Thus, t_c is found from the implicit relation

$$1 = \int_0^{t_c} d\tau\, e^{-\tau + \gamma\tau^2/2}. \tag{42}$$

In the limiting situations of small and large birth rate one has

$$t_c \sim \begin{cases} \ln(1/\gamma), & \gamma \ll 1; \\ \sqrt{\ln\gamma/\gamma}, & \gamma \gg 1. \end{cases} \tag{43}$$

Comparing to Eq. (37), it is seen that the parameter γ plays a similar role to v_0. Furthermore, in the vicinity of complete coverage, $1 - t/t_c \ll 1$, the gap distribution function approaches

$$f(x,t) \simeq (\gamma t_c)^2 (t_c - t) e^{-\gamma t_c x}, \tag{44}$$

which is surprisingly similar to equation (38). The uncovered fraction is simply $S(t) = t_c - t$ as for the case of simultaneous nucleation. Hence, the near-critical behavior is robust, *i.e.*, the details of the nucleation are not important.

C. Generalized Accelerated Nucleation

We turn now to the general accelerated nucleation-and-growth model with growth rate proportional to a power of the island length. Let us assume that an x-island grows with rate $(1+x)^\alpha$. Constant and linear growth rates correspond to $\alpha = 0$ and $\alpha = 1$, respectively.

Consider the case of simultaneous nucleation. We first write an *exact* rate equation satisfied by uncovered fraction:

$$\frac{dS(t)}{dt} = -N(t)\langle(1+x)^\alpha\rangle. \tag{45}$$

Remember that the previously investigated extreme cases of $\alpha = 0$ and $\alpha = 1$, yield $S(t) = N(t)$ for simultaneous nucleation (the initial density of seeds is set equal to one). Physically, it means that during the evolution process, the average size of inter-island gaps does not change. We now *assume* this feature for generalized nucleation-and-growth models as well. To proceed, we need to know the $\alpha^{\rm th}$ moment $\langle(1+x)^\alpha\rangle$. We certainly know the first moment $\langle x(t)\rangle = \big(1 - S(t)\big)/N(t)$. For $\alpha > 0$, we shall estimate $\langle(1+x)^\alpha\rangle$ by $[1 + \langle x\rangle]^\alpha$ which is exact only for $\alpha = 0$ and $\alpha = 1$. Inserting $S(t) = N(t)$ and $\langle(1+x)^\alpha\rangle \approx [1 + (1-S)/N]^\alpha = S^{-\alpha}$ into Eq. (45) we arrive at the *approximate* coverage rate equation

$$\frac{dS}{dt} = -S^{1-\alpha}, \tag{46}$$

which is solved to yield

$$S(t) = (1 - \alpha t)^{1/\alpha}. \tag{47}$$

Although there is no reason to expect that Eq. (47) provides an exact quantitative description for α other than 0 or 1, we do expect that complete coverage is still reached in a finite time and that the near-critical behavior is described by the exponent $1/\alpha$.

IV. SUMMARY

We investigated analytically nucleation-and-growth processes on one-dimensional substrates. We examined both constant and size dependent growth mechanisms. In the case of simultaneous nucleation the island size distribution has an infinite set of progressively weaker discontinuous spatial derivatives. We introduced the joint island-number density and showed that while exponential decay characterizes the fraction of islands containing a fixed number of seeds for simultaneous nucleation, a much slower power-law decay occurs for continuous nucleation. We generalized the KAJM theory to cases where the island growth velocity is linear in its size. Such an accelerated growth mechanism was shown to give rise to covering in a finite time, and the near-critical behavior of the system was found to be insensitive to most details of the growth process.

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