Impurity in a Maxwellian Unforced Granular Fluid

E. Ben-Naim[1] and P. L. Krapivsky[2]
[1] *Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*
[2] *Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215, USA*

We investigate velocity statistics of an impurity immersed in a uniform granular fluid. We consider the cooling phase, and obtain scaling solutions of the inelastic Maxwell model analytically. First, we analyze identical fluid-fluid and fluid-impurity collision rates. We show that light impurities have similar velocity statistics as the fluid background, although their temperature is generally different. Asymptotically, the temperature ratio increases with the impurity mass, and it diverges at some critical mass. Impurities heavier than this critical mass essentially scatter of a static fluid background. We then analyze an improved inelastic Maxwell model with collision rates that are proportional to the *average* fluid-fluid and fluid-impurity relative velocities. Here, the temperature ratio remains finite, and the system is always in the light impurity phase. Nevertheless, ratios of sufficiently high order moments $\langle v^n_{\rm impurity}\rangle/\langle v^n_{\rm fluid}\rangle$ may diverge, a consequence of the multiscaling asymptotic behavior.
PACS. 05.20.Dd, 02.50.-r, 47.70.Nd, 45.70.Mg

I. INTRODUCTION

Granular media are typically polydisperse. For example, sand and grains have a broad range of particle sizes and shapes. Such granular mixtures exhibit size segregation, a ubiquitous collective phenomena that underlies diverse processes including for example production and transport of powders in industry, sand dunes propagation and volcanic flows in geophysics [1,2]. The "Brazil Nut" problem where an impurity is immersed in a uniform granular media is an extreme realization of a granular mixture as it corresponds to the vanishing volume fraction limit of a binary mixture. While this problem has been extensively studied, dynamics of such impurities are not fully understood [3–10].

Understanding the velocity statistics of granular mixtures is a necessary step in describing multiphase granular flows. Experimental and theoretical studies show that in general, different components of a granular mixture are characterized by different typical speeds, i.e., granular temperatures are usually distinct [11]. However, analytical treatment of this case is difficult given the coupling between the different components of the mixture. An additional complication arises from the large number of parameters, including several restitution coefficients, governing the dynamics.

In contrast, the impurity problem is more amenable to analytical treatment. First, the impurity is directly enslaved to the fluid and second, there are fewer parameters [12]. In this paper, we study the impurity problem in the framework of the Maxwell model [13,14] that assumes that the collision rate is uniform, i.e., independent of the relative velocity of the colliding particles. This simplifies the Boltzmann collision operator and thence this model is widely used in kinetic theory [14–16]. The Maxwell model is analytically tractable even when the collisions are inelastic as shown in a number of recent studies of uniform and polydisperse granular gases [17–24].

We obtain analytic results for the velocity distributions valid for arbitrary spatial dimension and collision parameters. We consider two versions of the Maxwell model. In the first, termed the Inelastic Maxwell Model (IMM), the collision rates are completely independent of the relative velocities of the colliding particles. In the second, termed the Improved Inelastic Maxwell Model (IIMM), the collision rates are proportional to the *average* relative velocity of the colliding particles.

In the IMM, there are two phases separated by a critical impurity mass $m_*(r_p, r_q)$, determined by the restitution coefficients r_p and r_q characterizing fluid-fluid and fluid-impurity particle collisions, respectively. When the impurity mass m is smaller than the critical mass, $m < m_*$, different temperatures characterize the fluid and the impurity. The impurity velocity distribution exhibits similar characteristics as does the fluid; in particular, the same exponent governs the algebraic decay of the large velocity tail. Asymptotically, the impurity to fluid temperature ratio diverges as the critical mass is approached and is infinite when $m \geq m_*$. In this regime, the impurity essentially scatters of an ensemble of static fluid particles. The governing Lorentz-Boltzmann equation simplifies considerably. In general, moments of the velocity distribution exhibit multiscaling asymptotic behavior and the velocity distribution consists of replicas of the initial conditions. On the other hand, this phase transition is suppressed in the IIMM. The temperature ratio remains finite and the system is always in the light impurity phase. Nevertheless, secondary transitions affecting large velocity moments remain.

The rest of this paper is organized as follows. In Sec. II, we describe the IMM and present the Boltzmann equation for the fluid velocity distribution and the Lorentz-Boltzmann equation for the impurity velocity distribution. We then determine the temperature and show that a phase transition occurs. The light impurity phase is analyzed first and then the heavy impurity phase, emphasizing the behavior of the velocity distribution and its moments. In Sec. III, we discuss the IIMM and show that the system always remains in the light impurity phase. We conclude with a summary.

II. THE INELASTIC MAXWELL MODEL

We study dynamics of a single impurity particle in a uniform background of identical inelastic spheres. Without loss of generality, we set the mass of the fluid particles to unity, while the mass of the impurity is denoted by m. Particles interact via binary collisions that lead to exchange of momentum along the impact direction. Collisions between two fluid particles are characterized by the collision parameter p, and collisions between the impurity and any fluid particle are characterized by the collision parameter q. When a particle of velocity $\mathbf{u}_1$ collides with a fluid particle of velocity $\mathbf{u}_2$ its post-collision velocity $\mathbf{v}_1$ is given by

$$\mathbf{v}_1 = \mathbf{u}_1 - (1-p)\,(\mathbf{g}\cdot\mathbf{n})\,\mathbf{n}, \tag{1}$$
$$\mathbf{v}_1 = \mathbf{u}_1 - (1-q)\,(\mathbf{g}\cdot\mathbf{n})\,\mathbf{n}, \tag{2}$$

with $\mathbf{g}=\mathbf{u}_1-\mathbf{u}_2$ the relative velocity and $\mathbf{n}$ the unit vector parallel to the impact direction. The first equation gives the velocity of a fluid particle and the second that of an impurity. The collision rules (1)–(2) are derived by employing momentum conservation combined with the fact that in an inelastic collision, the component of the relative velocity parallel to the impact direction is reduced by a factor equal to the fluid-fluid (impurity-fluid) restitution coefficient r_p (r_q). The restitution coefficients are related to the collision parameters via

$$r_p = 1-2p, \tag{3}$$
$$r_q = m-(m+1)\,q. \tag{4}$$

Since the restitution coefficients obey $0\le r\le 1$, the collision parameters satisfy $0\le p\le 1/2$ and $\frac{m-1}{m+1}\le q\le\frac{m}{m+1}$. The energy dissipated in each collision is equal to $A(\mathbf{g}\cdot\mathbf{n})^2$, with $A=2p(1-p)$ and $A=m(1-q)[2-(m+1)(1-q)]$ for fluid-fluid and impurity-fluid collisions, respectively.

We consider a collision process where random pairs of particles undergo inelastic collisions with a random impact direction. This inelastic Maxwell model (IMM) is described by a Boltzmann equation with a uniform collision rate. The model is a straightforward generalization of the classical Maxwell model. Specifically, Maxwell showed [13] that for elastic "Maxwell molecules" interacting via a repulsive $r^{-2(d-1)}$ potential in d spatial dimensions, the collision rate does not depend on the magnitude of the relative velocity.

Let $P(\mathbf{v},t)$ and $Q(\mathbf{v},t)$ be the normalized velocity distributions of the background and the impurity, respectively. In this collision process, no correlations develop and the governing Boltzmann equation and the Lorentz-Boltzmann equation are

$$\begin{aligned}\frac{\partial P(\mathbf{v},t)}{\partial t} &= \int d\mathbf{n}\int d\mathbf{u}_1\,P(\mathbf{u}_1,t)\int d\mathbf{u}_2\,P(\mathbf{u}_2,t) \\ &\times\left\{\delta\Big[\mathbf{v}-\mathbf{u}_1+(1-p)(\mathbf{g}\cdot\mathbf{n})\mathbf{n}\Big]-\delta(\mathbf{v}-\mathbf{u}_1)\right\},\end{aligned} \tag{5}$$

$$\begin{aligned}\frac{\partial Q(\mathbf{v},t)}{\partial t} &= \int d\mathbf{n}\int d\mathbf{u}_1\,Q(\mathbf{u}_1,t)\int d\mathbf{u}_2\,P(\mathbf{u}_2,t) \\ &\times\left\{\delta\Big[\mathbf{v}-\mathbf{u}_1+(1-q)(\mathbf{g}\cdot\mathbf{n})\mathbf{n}\Big]-\delta(\mathbf{v}-\mathbf{u}_1)\right\}.\end{aligned} \tag{6}$$

The angular integration over the impact direction is normalized to unity, $\int d\mathbf{n}=1$. The collision integrals directly reflect the collision rules.

In writing Eqs. (5)–(6), fluid-fluid and impurity-fluid collision rates were assumed to be the same and were set to unity for convenience. Hence the average number of collisions experienced by a particle equals time. The results detailed in this section are exact for a mean-field collision process where, irrespective of their type, random pairs of particles are chosen to undergo inelastic collisions according to (1)–(2). A more realistic collision rates model is treated in Sec. III.

The Boltzmann and Lorentz-Boltzmann equations simplify in Fourier space. Thence we use the Fourier transforms of the velocity distribution functions,

$$F(\mathbf{k},t) = \int d\mathbf{v}\,e^{i\mathbf{k}\cdot\mathbf{v}}\,P(\mathbf{v},t), \tag{7}$$
$$G(\mathbf{k},t) = \int d\mathbf{v}\,e^{i\mathbf{k}\cdot\mathbf{v}}\,Q(\mathbf{v},t). \tag{8}$$

The convolution structure of Eqs. (5)–(6) implies that the collision terms factorize

$$\frac{\partial}{\partial t}F(\mathbf{k},t)+F(\mathbf{k},t) = \int d\mathbf{n}\,F\left[\mathbf{k}-\mathbf{p},t\right]F\left[\mathbf{p},t\right], \tag{9}$$
$$\frac{\partial}{\partial t}G(\mathbf{k},t)+G(\mathbf{k},t) = \int d\mathbf{n}\,G\left[\mathbf{k}-\mathbf{q},t\right]F\left[\mathbf{q},t\right], \tag{10}$$

with $\mathbf{p}=(1-p)\,(\mathbf{k}\cdot\mathbf{n})\,\mathbf{n}$ and $\mathbf{q}=(1-q)\,(\mathbf{k}\cdot\mathbf{n})\,\mathbf{n}$. These equations reflect the momentum transfer occurring during collisions. They also hint that the model is analytically tractable. For example, expansion in powers of the wave number shows that moments of the velocity distributions obey closed hierarchies of evolution equations.

A. The Temperature

We study the freely evolving case where in the absence of energy input the system "cools" indefinitely [25,26]. The fluid temperature is defined as the average square of the velocity: $T=\frac{1}{d}\int d\mathbf{v}\,v^2P(\mathbf{v},t)$ with $v\equiv|\mathbf{v}|$. From the Boltzmann equation (5), the fluid temperature evolves according to $\frac{d}{dt}T=-\lambda T$ with $\lambda=2p(1-p)\int d\mathbf{n}\,n_1^2$. The angular integration is readily performed, $\int d\mathbf{n}\,n_1^2=1/d$, by using symmetry and the identity $n_1^2+\ldots+n_d^2=1$. Hence the fluid temperature satisfies

$$\frac{d}{dt}T = -\frac{2p(1-p)}{d}\,T, \tag{11}$$

implying that the fluid temperature decays exponentially with time

$$T(t) = T_0\, e^{-2p(1-p)t/d}. \tag{12}$$

Similarly, from the Lorentz-Boltzmann equation (6) one can obtain the governing equation for the impurity temperature $\Theta(t) = \frac{1}{d}\int d\mathbf{v}\, v^2 Q(\mathbf{v},t)$. One finds that the impurity temperature is coupled to the fluid temperature via a simple linear rate equation

$$\frac{d}{dt}\Theta = -\frac{1-q^2}{d}\,\Theta + \frac{(1-q)^2}{d}\,T. \tag{13}$$

The solution to this equation is a linear combination of two exponentials

$$\Theta(t) = (\Theta_0 - c\,T_0)\; e^{-(1-q^2)t/d} + c\,T_0\, e^{-2p(1-p)t/d}, \tag{14}$$

with the constant

$$c = \frac{(1-q)^2}{1-q^2-2p(1-p)}. \tag{15}$$

Therefore, there are two different regimes of behavior. When $1-q^2 > 2p(1-p)$, the impurity temperature is proportional to the fluid temperature asymptotically, $\frac{\Theta(t)}{T(t)} \to c$ as $t \to \infty$. In the complementary region $2p(1-p) > 1-q^2$ an extreme violation of equipartition occurs, as the ratio of the fluid temperature to the impurity temperature vanishes. The impurity is very energetic compared with the fluid and it practically sees a static fluid. From Eq. (15) we find that at the transition point $q = \sqrt{1-2p(1-p)}$. Employing relations (3)–(4) between the restitution coefficients and the collision parameters we obtain the critical mass

$$m_* = \frac{r_q + \sqrt{(1+r_p^2)/2}}{1-\sqrt{(1+r_p^2)/2}}. \tag{16}$$

The heavy impurity phase arises when the impurity is a bit heavier than the fluid particles: Even when the fluid-fluid collisions are completely inelastic ($r_p = 0$), the critical mass satisfies $m_* > 1+\sqrt{2}$. For weakly dissipative fluids ($r \to 1$), the critical mass diverges, $m_* \propto (1-r_p)^{-1}$ (see Fig. 1).

Note now a few features of the light impurity phase. First, Eq. (15) generalizes the elastic fluid ($p = 0$) result $c = (1-q)/(1+q)$ [27,28]. That result was actually established for a hard sphere fluid, so at least asymptotically both the IMM and the improved model predict the same impurity temperature when the fluid is elastic. Further, the initial impurity temperature becomes irrelevant and the impurity is governed by the fluid background. The average energies of the impurity and fluid particles are asymptotically equal when $m\Theta/T \to 1$, or when $mc = 1$. Thus, energy equipartition occurs on a particular surface in the three dimensional space (m, r_p, r_q) where $m(1-q)^2 = 1-q^2-2p(1-p)$. Using relations (3)–(4) between the collision parameters and the restitution coefficients, energy equipartition occurs when the impurity mass $m_{\rm eq}$ is given by

$$m_{\rm eq} = \frac{1+r_p^2-2r_q^2}{1-r_q^2}. \tag{17}$$

As expected, this mass equals unity when $r_p = r_q$. Curiously, $m_{\rm eq}$ vanishes when $r_q^2 = (1+r_p^2)/2$, indicating that for $r_q > \sqrt{(1+r_p^2)/2}$, energy equipartition does not occur, as shown in Fig. 2. However, in a generic point in the parameter space (m, r_p, r_q) equipartition does break down [11,23]. This is a signature of the dissipative and nonequilibrium nature of the system.

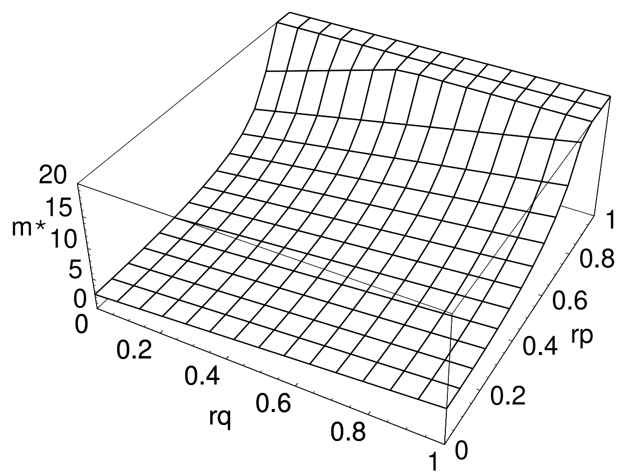

FIG. 1. The critical mass m_* versus the restitution coefficients r_p and r_q.

In summary, when the impurity mass is smaller than the critical mass $m < m_*$, the fluid temperature governs the impurity temperature. Otherwise, the impurity is infinitely more energetic than the fluid asymptotically. As $t \to \infty$ one finds

$$\frac{\Theta(t)}{T(t)} \to \begin{cases} c & m < m_*, \\ \infty & m \geq m_*. \end{cases} \tag{18}$$

At the critical impurity mass, $m = m_*$, the solution to Eq. (13) shows that the temperature ratio diverges linearly with time, $\frac{\Theta(t)}{T(t)} \to 2p(1-p)\,t$. Interestingly, the dependence on the dimension is secondary as it only sets the overall time scale (the transformation $t \to t/d$ absorbs the dimension dependence). For example, both the critical mass, m_*, and the temperature ratio c are independent of d.

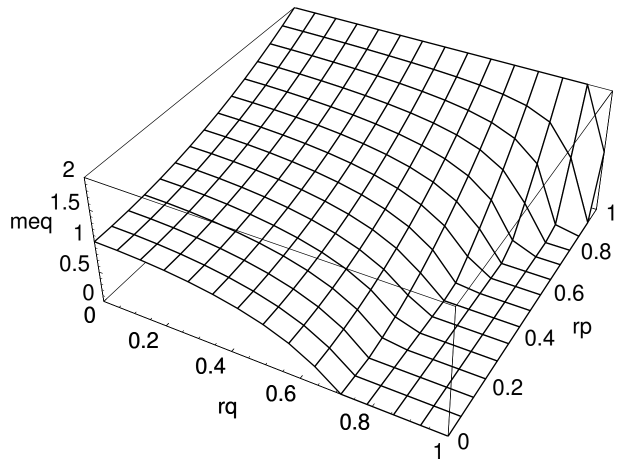

FIG. 2. The equipartition mass $m_{\rm eq}$, given by Eq. (17), versus the restitution coefficients r_p and r_q.

B. The light impurity phase

For a light impurity, $m < m_*$, the fluid velocity statistics govern basic characteristics of the impurity statistics. We have seen this for the temperature, as the initial impurity temperature is irrelevant asymptotically, and the ratio of the two temperatures approaches a constant $\frac{\Theta(t)}{T(t)} \to c$. The question arises how more detailed characteristics such as the impurity velocity distribution emerge from the fluid velocity distribution. This question can be answered in detail in one dimension where the explicit scaling solution for the fluid velocity distribution is known [20]. Below we primarily focus on the one-dimensional case and derive an explicit exact solution. We then briefly comment on the higher-dimensional case, where the solution is very cumbersome.

The velocity distribution of the fluid approaches a scaling form asymptotically, $P(v,t) \to T^{-1/2}\mathcal{P}\left(vT^{-1/2}\right)$. Following the earlier treatment of the fluid velocity distribution [17,20–22], we employ the Fourier transform technique. The fluid Fourier transform is thus characterized by the scaling form $F(k,t) = f\left(|k|\,T^{1/2}\right)$. The governing equation [17] for the corresponding scaling function is obtained from (9) to give

$$-p(1-p)zf'(z) + f(z) = f(pz)f(z-pz), \tag{19}$$

and the solution is $f(z) = (1+z)\,e^{-z}$ [20].

The behavior of the fluid suggests that the impurity velocity distribution also approaches a scaling form $Q(v,t) \to T^{-1/2}\mathcal{Q}\left(vT^{-1/2}\right)$. Then the corresponding Fourier transform reads $G(k,t) = g\left(|k|\,T^{1/2}\right)$. From the Lorentz-Boltzmann equation (10), this scaling function satisfies the *linear* equation

$$-p(1-p)zg'(z) + g(z) = (1+az)\,e^{-az}\,g(qz) \tag{20}$$

with $a = 1-q$. The fluid scaling function is a combination of $z^n e^{-z}$ with $n=0$ and $n=1$. We therefore seek a similar solution of Eq. (20)

$$g(z) = \sum_{n=0}^{\infty} A_n z^n e^{-z}. \tag{21}$$

Inserting (21) into (20), the exponential factors cancel. Equating terms of the order z^n, yields the following recursion relation for the coefficients:

$$A_n = \frac{q^{n-1}(1-q) - p(1-p)}{1 - q^n - np(1-p)}\,A_{n-1}. \tag{22}$$

From the normalization $g(0) = 1$ we get $A_0 = 1$. The next two coefficients are $A_1 = 1$, $A_2 = (1-c)/2$. Using these values, one computes the small z expansion of the solution (21): $g(z) = 1 - cz^2/2 + \mathcal{O}(z^3)$, and the first non-trivial coefficient is indeed consistent the definition of the Fourier transform.

The Fourier transform can be inverted to obtain the impurity velocity distribution function explicitly. The inverse Fourier transform of $e^{-\kappa z}$ is $\frac{1}{\pi}\frac{\kappa}{\kappa^2+w^2}$; the inverse transforms of $z^n e^{-z}$ can be obtained using successive differentiation with respect to κ. In general, the velocity scaling function is

$$\mathcal{Q}(w) = \frac{1}{\pi}\sum_{n=0}^{\infty} A_n (-1)^n \frac{d^n}{d\kappa^n}\frac{\kappa}{\kappa^2+w^2}\bigg|_{\kappa=1}. \tag{23}$$

In other words, one can express the solution as a combination of powers of Lorentzians

$$\mathcal{Q}(w) = \frac{2}{\pi}\sum_{n=2}^{\infty} B_n \left(\frac{1}{1+w^2}\right)^n. \tag{24}$$

The coefficients B_n are linear combinations of the coefficients A_k's with $k \leq n+1$, e.g., $B_2 = 1 - 3A_2 + 3A_3$ and $B_3 = 4A_2 - 24A_3 + 60A_4$. The first squared Lorentzian term dominates the tail of the velocity distribution $\mathcal{Q}(w) \sim \mathcal{P}(w) \sim w^{-4}$, as $w \to \infty$. This algebraic behavior prevails for $w \gg 1$. Therefore, the impurity has the same extremal velocity statistics as the fluid.

An interesting aspect of this solution is that for particular values of the collision parameter q, the infinite sum terminates at a finite order. Of course, when $q_1 = p$ then

$$\mathcal{Q}(w) = \mathcal{P}(w) = \frac{2}{\pi}\frac{1}{(1+w^2)^2}. \tag{25}$$

This is reflected by the vanishing coefficient $A_2 = 0$. The multiplicative recursion relation (22) then shows that $A_n = 0$ for all $n \geq 2$. Furthermore, for $q_2^2(1-q_2) = p(1-p)$, one has $A_3 = 0$, and the velocity distribution (24) contains only two terms

$$\mathcal{Q}(w) = \frac{2}{\pi}\left[\frac{1-3A_2}{\left(1+w^2\right)^2} + \frac{4A_2}{\left(1+w^2\right)^3}\right], \tag{26}$$

with $A_2 = q/(1+2q)$. Similarly, the solution can be a finite sum with k terms when p and q_k are related via

$q_k^k(1-q_k)=p(1-p)$ thereby imposing $A_n=0$ for $n>k$. For given fluid collision parameter p, the above relation has the solution q_k only for sufficiently small k. Thus, there are a few special values of the impurity collision parameter q_k for which the scaled impurity velocity distribution is a linear combination of simple rational functions $\left(1+w^2\right)^{-n}$ with $n=2,\ldots,k+1$.

While the scaling functions underlying the impurity and the fluid are similar, more subtle features may differ. In particular, the full time dependent behavior, as characterized by the moments of the impurity distribution, exhibits rich behavior. Let $L_n(t)=\int dv\, v^n\, P(v,t)$ be the moments of the fluid velocity distribution. Multiplying the equations (5)–(6) by v^n and integrating, the moments obey the recursive equations

$$\frac{d}{dt}L_n+a_nL_n=\sum_{j=2}^{n-2}\binom{n}{j}p^j(1-p)^{n-j}L_jL_{n-j}, \qquad (27)$$

$$\frac{d}{dt}M_n+b_nM_n=\sum_{j=0}^{n-2}\binom{n}{j}q^j(1-q)^{n-j}M_jL_{n-j}, \qquad (28)$$

with

$$a_n(p)=1-p^n-(1-p)^n, \qquad b_n(q)=1-q^n. \qquad (29)$$

Asymptotically, the fluid moments decay exponentially according to [17]

$$L_n(t)\propto e^{-a_n(p)\,t}. \qquad (30)$$

Using this asymptotics we analyze the behavior of the impurity moments. The second moment, i.e. the impurity temperature, was already shown to behave similar to the fluid temperature when $a_2(p)<b_2(q)$. The fourth moment behaves similarly to the fourth moment of the fluid when $a_4(p)<b_4(q)$, and generally the first n moments are proportional to each other, $M_2\propto L_2,\ldots,M_{2n}\propto L_{2n}$, when $a_{2k}(p)<b_{2k}(q)$ for $k=1,\ldots,n$. In particular, *all* respective impurity and fluid moments are proportional to each other when $a_{2k}(p)<b_{2k}(q)$ is valid for every k. It is easy to see that above inequalities hold when $p+q<1$, which is always obeyed when $m<1$.

However, in the complementary parameter range, $a_n(p)>b_n(q)$ for sufficiently large n. Such moments are no longer governed by the fluid and

$$M_n(t)\propto e^{-(1-q^n)t}. \qquad (31)$$

The corresponding moment ratio diverges asymptotically: $M_n/L_n\to\infty$ as $t\to\infty$. Interestingly, the same behavior (31) is found in the heavy impurity phase, as will be shown below. Therefore, the two phases are not entirely distinct. A series of transitions affecting moments of decreasing order occurring at increasing masses,

$$m_1>m_2>\cdots>m_\infty, \qquad (32)$$

signals the transition to the heavy impurity phase. When $m\geq m_n$, the ratio M_{2k}/L_{2k} diverges asymptotically for all $k\geq n$. This generalizes the second moment transition occurring at $m_1\equiv m_*$. The transition masses

$$m_n=\frac{r_q+\frac{1}{2}\left[(1-r_p)^{2n}+(1+r_p)^{2n}\right]^{\frac{1}{2n}}}{1-\frac{1}{2}\left[(1-r_p)^{2n}+(1+r_p)^{2n}\right]^{\frac{1}{2n}}} \qquad (33)$$

are found from $q^{2n}=p^{2n}+(1-p)^{2n}$ and Eq. (4). In particular, $m_\infty=\lim_{n\to\infty}m_n=(1+r_p+2r_q)/(1-r_p)$, and sufficiently light impurities ($m<m_\infty$) mimic the fluid completely. Additionally, since $m_n>m_\infty\geq 1$, the impurity must be heavier than the fluid for any transition to occur.

The above analysis of the light impurity phase suggests that in higher dimensions, the impurity velocity distribution might be similar to that of the fluid. In higher dimensions we again assume that the Lorentz-Boltzmann equation admits a scaling solution. The corresponding equation in Fourier space (10) then considerably simplifies. Following the earlier treatments of the fluid case [21,22] we extract the high-energy tail from the small-k behavior of the Fourier transform of the scaled velocity distribution. The outcome is that both velocity distributions have the same algebraic high-energy tail

$$\mathcal{Q}(w)\sim\mathcal{P}(w)\sim w^{-\sigma}, \qquad (34)$$

with the exponent σ calculated in [21,22]. Such analysis also yields the ratio of the prefactors governing this algebraic decay.

C. The heavy impurity phase

When the mass of the impurity is equal to or larger than the critical mass, $m\geq m_*$, the velocities of the fluid particles are asymptotically negligible compared with the velocity of the impurity. Hence, in the $t\to\infty$ limit, fluid particles become stationary as viewed by the impurity. Therefore, one can set $\mathbf{u}_2\equiv 0$ in the collision rule (2):

$$\mathbf{v}=\mathbf{u}-(1-q)\,(\mathbf{u}\cdot\mathbf{n})\,\mathbf{n}. \qquad (35)$$

This process is somewhat analogous to a Lorentz gas [29]. However, in the granular impurity system, a heavy particle scatters of a static background of lighter particles, while in the Lorentz gas the scatterers are infinitely massive. Despite this difference, the mathematical descriptions of the two problems are similar. Specifically, the collision rule for the inelastic Lorentz gas [27,28] is obtained from (35) by a mere replacement of the factor $(1-q)$ with $(1+r_q)$.

Let us first consider the one-dimensional case where an explicit solution of the velocity distribution is possible. Setting $u_2\equiv 0$ in the delta function in the Lorentz-Boltzmann equation (6), integration over the fluid velocity u_2 is trivial, $\int du_2\, P(u_2,t)=1$, and integration over the impurity velocity u_1 gives

$$\frac{\partial}{\partial t}Q(v,t)+Q(v,t)=\frac{1}{q}Q\left(\frac{v}{q}\,,\,t\right). \tag{36}$$

This equation can be solved directly by considering the stochastic process the impurity particle experiences. In a sequence of collisions, the impurity velocity changes according to $v_0 \to qv_0 \to q^2 v_0 \to \cdots$ with v_0 the initial velocity. After n collisions the impurity velocity decreases exponentially, $v_n = q^n v_0$. The collision rate is unity, and hence, the average number of collisions experienced till time t equals t. Furthermore, the collision process is random, and therefore, the probability that the impurity undergoes exactly n collisions up to time t is Poissonian $t^n e^{-t}/n!$. Thus, the velocity distribution function reads

$$Q(v,t)=e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}\,\frac{1}{q^n}\,Q_0\left(\frac{v}{q^n}\right), \tag{37}$$

where $Q_0(v)$ is the initial velocity distribution of the impurity. Indeed, one can check that this properly normalized solution satisfies Eq. (36).

Interestingly, the impurity velocity distribution function is a time-dependent combination of "replicas" of the initial velocity distribution. Since the corresponding argument is stretched, compact velocity distributions display an infinite set of singularities, a generic feature of the Maxwell model [15,17].

The impurity velocity distribution exhibits interesting asymptotic behaviors. Consider for simplicity the uniform initial velocity distribution: $Q_0(v)=1$ for $|v|<1/2$ and $Q_0(v)=0$ otherwise. The solution (37) reduces to a finite sum, with $n \le N = \frac{\ln(2v)}{\ln q}$. In the physically interesting limits $t\to\infty$ and $v\to 0$, the sum on the right-hand side of Eq. (37) simplifies to respectively $e^{t/q}$ or $(t/q)^N/N!$ when the number of terms is above or below the threshold value $N=t/q$. The magnitude of $Q(v,t)$ above the threshold greatly exceeds the magnitude below the threshold, so $Q(v,t)$ appears to approach a step function. A refined analysis shows that the width of the front widens diffusively so the front remains smooth although its relative width vanishes. Specifically, one finds the following traveling-wave like scaling solution

$$Q(v,t)\to e^{-t+t/q}\,\Psi(\eta), \tag{38}$$

with the following wave form and coordinate

$$\Psi(\eta)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\eta}dx\,e^{-x^2},\quad \eta=\frac{\frac{q}{\ln q}\,\ln(2v)-t}{\sqrt{2qt}}. \tag{39}$$

Note, however, that the large velocity tail ($\eta\to\infty$), ignored in Eq. (39), provides actually the dominant contribution to the moments. This is an unusual traveling wave form in the sense that the argument is the logarithm of the velocity rather then the velocity itself, a reflection of the exponentially decaying velocity.

In contrast to the velocity distribution, the moments $M_n(t)=\int dv\, v^n Q(v,t)$ exhibit a much simpler behavior. Indeed, from Eq. (36) one finds that every moment is coupled only to itself, $\frac{d}{dt}M_n = -(1-q^n)M_n$. Solving this equation we recover Eq. (31); in the heavy impurity phase, however, it holds for all n. Therefore the moments exhibit multiscaling asymptotic behavior. The decay coefficients, characterizing the n-th moment, depend on n in a nonlinear fashion. This multiscaling behavior excludes scaling solutions with sharp tails (stretched exponentials decays and faster). While collisions with the fluid are sub-dominant, they still lead to corrections to the leading asymptotic behavior.

We turn now to arbitrary spatial dimensions d. The impurity velocity changes according to Eq. (35) with the impact direction $\mathbf{n}$ chosen randomly. The corresponding Lorentz-Boltzmann equation reads

$$\frac{\partial}{\partial t}Q(\mathbf{v},t)+Q(\mathbf{v},t)=\int d\mathbf{n}\int d\mathbf{u}\,Q(\mathbf{u},t)\times\delta\Big[\mathbf{v}-\mathbf{u}+(1-q)(\mathbf{u}\cdot\mathbf{n})\,\mathbf{n}\Big]. \tag{40}$$

Moments of the velocity distribution

$$M_n(t)=\int d\mathbf{v}\,v^n\,Q(\mathbf{v},t), \tag{41}$$

can be obtained directly. We focus on the even moments of the distribution. Indeed, they satisfy the following evolution equation

$$\frac{d}{dt}M_{2n}=-\big(1-\langle\xi^n\rangle\big)M_{2n} \tag{42}$$

where $\mu=\cos^2\theta=(\hat{\mathbf{u}}\cdot\mathbf{n})^2$, $\xi\equiv\xi(q,\mu)=1-(1-q^2)\mu$, and $\langle\cdot\rangle$ is the shorthand notation for the angular integration: $\langle\, f\,\rangle\equiv\int_0^1\mathcal{D}\mu\, f(\mu)$. Since $d\mathbf{n}\propto\sin^{n-2}\theta\,d\theta$, the (normalized) integration measure $\mathcal{D}\mu$ is

$$B\left(\frac{1}{2},\frac{d-1}{2}\right)\mathcal{D}\mu=\mu^{-\frac{1}{2}}(1-\mu)^{\frac{d-3}{2}}d\mu \tag{43}$$

where $B(a,b)$ is the beta function. For example, $\langle 1\rangle=1$, and $\langle\mu\rangle=1/d$.

From the evolution equations (42), the moments are found to decay exponentially with time

$$M_{2n}(t)=M_{2n}(0)\,e^{-(1-\langle\xi^n\rangle)t}. \tag{44}$$

In particular, $1-\langle\xi\rangle=(1-q^2)/d$, and thus, the temperature decay $\Theta(t)=\Theta(0)\,e^{-(1-q^2)t/d}$ is recovered. There is a little discrepancy with the exact temperature of Eq. (14) which depends on the initial fluid temperature, T_0. This is a remnant of a transient regime where the two velocity scales are comparable. Generally, the time dependence is correct. Note also that in one dimension $\xi=q^2$ and hence Eq. (44) reduces to our earlier result. In the infinite dimension limit, $\mu\to 0$, and all moments decay according to e^{-t}. However, in general, the moments exhibit multiscaling asymptotic behavior, and knowledge of the typical velocity is insufficient to fully characterize the entire velocity distribution. Indeed, writing

$M_{2n} \sim M_2^{\alpha_n}$, the exponents $\alpha_n = (1 - \langle \xi^n \rangle)/(1 - \langle \xi \rangle)$ have a nontrivial spectrum.

The moments directly give a formal exact solution of the Fourier transform of the impurity velocity distribution (8). We consider isotropic situations where $G(\mathbf{k}, t) \equiv G(k^2, t)$ with $k \equiv |\mathbf{k}|$. Expanding the transform in powers of k^2 and substituting the moment result (44) yields

$$G(\mathbf{k}, t) = e^{-t} \sum_{n=0}^{\infty} \frac{(-k^2)^n \langle \mu^n \rangle}{(2n)!} M_{2n}(0)\, e^{\langle \xi^n \rangle t}. \qquad (45)$$

While this is an explicit solution, it is not too illuminating. First, it is in Fourier space, and second, it involves the complicated angular averages $\langle \xi^n \rangle$.

Nevertheless, it can be shown that the solution remains a time dependent combination of properly modified replicas of the initial distribution. Indeed, either from Eq. (45) or directly from the Fourier transform equation $\frac{\partial}{\partial t} G(\mathbf{k}, t) + G(\mathbf{k}, t) = \langle G(\mathbf{k}\xi, t) \rangle$, the solution can be rewritten in the form

$$G(k^2, t) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} G_n(k^2). \qquad (46)$$

Here, $G_0(k^2)$ is the initial Fourier transform, and the "building blocks" G_n are obtained from a recursive procedure of angular integration $G_{n+1}(k^2) = \langle G_n(k^2 \xi) \rangle$.

III. THE IMPROVED INELASTIC MAXWELL MODEL

In the IMM, the rates for fluid-fluid and fluid-impurity collisions were identical (and therefore, set to unity for convenience). For granular fluids, however, the collision rate is proportional to the relative velocity [30–33]. Therefore, one can improve the Maxwell model by replacing the actual collision rate with an *average* collision rate proportional to the average relative velocity. The simplest choice of the average relative velocity is $\sqrt{\langle (\mathbf{v}_1 - \mathbf{v}_2)^2 \rangle} \propto \sqrt{(T_1 + T_2)/2}$. For fluid-fluid and impurity-fluid collisions we thus obtain $\sqrt{T}$ and $\sqrt{(T + \Theta)/2}$, respectively. Thus, different collision rates multiply the collision integrals in the Boltzmann and Lorentz-Boltzmann equations (5)–(6). In the fluid case, this overall prefactor merely affects the time dependence of the temperature. As will be shown below, in the impurity case, the above phase transition is suppressed in the IIMM, although secondary transitions corresponding to higher order moments remain.

Let us again start with the behavior of the temperature. The fluid temperature satisfies

$$\frac{d}{dt} T = -\sqrt{T} \left[\frac{2p(1-p)}{d} T \right]. \qquad (47)$$

Solving this equation, we recover Haff's cooling law $T(t) = T_0[1 + t/t_0]^{-2}$, with T_0 the initial temperature and $t_0 = d/[p(1-p)T_0^{1/2}]$ [25].

The corresponding rate equation for the impurity temperature Θ is

$$\frac{d}{dt} \Theta = \sqrt{\frac{T + \Theta}{2}} \left[-\frac{1 - q^2}{d} \Theta + \frac{(1-q)^2}{d} T \right]. \qquad (48)$$

Since we are primarily interested in the temperature ratio, $S = \Theta/T$, we study this quantity directly. It evolves according to

$$\frac{1}{\sqrt{T}} \frac{d}{dt} S = \sqrt{\frac{1 + S}{2}} \left[-\frac{1 - q^2}{d} S + \frac{(1-q)^2}{d} \right] + \frac{2p(1-p)}{d} S. \qquad (49)$$

In the inelastic Maxwell model, the gain and the loss terms were comparable, both increasing linearly with S. Here, in contrast, the loss term, which grows as $S^{3/2}$, eventually overtakes the gain term that grows only linearly with S. Therefore, $S \to c$ where c is the root of the cubic equation

$$\sqrt{\frac{1 + c}{2}} \left(c - \frac{1 - q}{1 + q} \right) = \frac{2p(1-p)}{1 - q^2} c. \qquad (50)$$

Consequently, there is only one phase, the light impurity phase. We note that the ratio c is independent of the spatial dimension d. Intuitively, since the impurity collision rate relatively increases with the impurity temperature, the impurity energy dissipation rate increases, thereby limiting the (relative) growth of the impurity temperature.

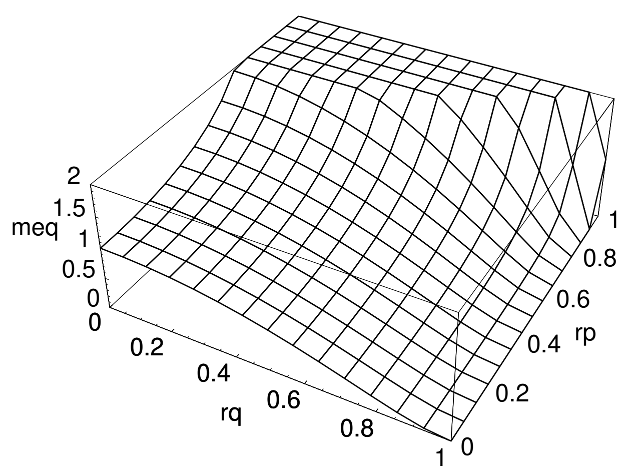

FIG. 3. The equipartition mass $m_{\rm eq}$, given by Eq. (51), versus the restitution coefficients r_p and r_q.

Generally, there is no equipartition of energy except for a particular surface in the space (m, r_p, r_q). Energy equipartition occurs when $mc = 1$. Using the relations (3)–(4), we find the equipartition mass

$$m_{\rm eq} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 8\left(\frac{1-r_q^2}{1-r_p^2}\right)^2}. \qquad (51)$$

Figure 3 plots $m_{\rm eq} = m_{\rm eq}(r_p, r_q)$.

Qualitatively, our findings in the light impurity phase of the IMM extend to the IIMM. For example, both velocity distributions follow scaling forms and the large-velocity tails of both distributions are the same. In the one dimensional case, explicit expressions for the impurity scaling function are possible, and as the treatment follows closely that outlined in the light impurity phase, we briefly outline the results. In 1D, the impurity velocity distribution approaches a scaling solution $Q(v,t) \to T^{-1/2}\mathcal{Q}\left(vT^{1/2}\right)$. The corresponding Fourier transform reads $G(k,\tau) = g\left(|k|\,T^{1/2}\right)$. The only difference with the above inelastic Maxwell model is that the collision terms are proportional to $\beta^{-1} = \sqrt{(1+c)/2}$. Consequently, Eq. (20) generalizes as follows

$$-\beta p(1-p) z g'(z) + g(z) = (1+az)\, e^{-az}\, g(qz). \qquad (52)$$

Seeking a series solution of the form (21), leads to the following recursion relations for the coefficients

$$A_n = \frac{(1-q)q^{n-1} - \beta p(1-p)}{1 - q^n - n\beta p(1-p)}\, A_{n-1}, \qquad (53)$$

with $A_0 = A_1 = 1$. Again, the velocity distribution is a combination of powers of Lorentzians as in Eq. (24):

$$\mathcal{Q}(w) = \frac{2}{\pi}\sum_{n=2}^{\infty} B_n \left(1+w^2\right)^{-n}, \qquad (54)$$

where B_n are linear combinations of the coefficients A_n's given by the same expressions as in Sec. II. In particular, the large-velocity tail is generic $\mathcal{Q}(w) \sim w^{-4}$.

Given the algebraic form of the velocity distributions, we examine the asymptotic behavior of moments of the velocity distribution. Moments of the fluid and the impurity, L_n and M_n, respectively, evolve according to a straightforward generalization of Eqs.(27)–(28),

$$\frac{d}{d\tau_p} L_n + a_n L_n = \sum_{j=2}^{n-2} \binom{n}{j} p^j (1-p)^{n-j} L_j L_{n-j}, \qquad (55)$$

$$\frac{d}{d\tau_q} M_n + b_n M_n = \sum_{j=0}^{n-2} \binom{n}{j} q^j (1-q)^{n-j} M_j L_{n-j}. \qquad (56)$$

Here $a_n(p)$ and $b_n(q)$ are given by Eq. (29) and the collision counters,

$$\tau_p = \int_0^t dt'\, \sqrt{T}, \qquad \tau_q = \int_0^t dt'\, \sqrt{(T+\Theta)/2},$$

play the role of time in Eq. (55) and (56), respectively. Since the two temperatures are asymptotically proportional to each other, $\tau_q \to \tau_p/\beta$. The fluid moments decay according to $L_n \propto e^{-a_n(p)\tau_p} \propto t^{-2a_n/a_2}$ [17]. Inserting the asymptotics $L_n \propto e^{-a_n(p)\beta\tau_q}$ into Eq. (56) and performing the same analysis as in the IMM we find that when $\beta a_n(p) < b_n(q)$, the impurity moments are enslaved to the fluid moments, i.e., $M_n \propto L_n$ asymptotically. Otherwise, sufficiently large impurity moments behave differently than the fluid moments, viz. $L_n \propto e^{-b_n\tau_q} \propto t^{-2b_n/\beta a_n}$. Although the primary transition affecting the second moment does not occur, secondary transitions affecting larger moments do occur at a series of masses, as in Eq. (32). In the IIMM $m_1 \equiv m_*$ diverges, but other masses remain finite. The transition masses m_n are found by solving $\beta a_{2n}(p) = b_{2n}(q)$ simultaneously with Eq. (50), and then applying Eq. (4). For example, for completely inelastic collisions ($r_p = r_q = 0$) one finds $m_2 = 1.65$. Qualitatively, these transitions imply that some velocity statistics of the impurity, specifically large moments, are no longer governed by the fluid.

IV. SUMMARY

We have studied dynamics of impurities in granular fluids using the inelastic Maxwell model. In general, there is breakdown of energy equipartition as two different temperatures characterize the impurity and the fluid. We analyzed two different models. First, we considered identical fluid-fluid and fluid-impurity overall collision rates. In this case a phase transition, marked by a dissipation dependent critical impurity mass, occurs. Breakdown of equipartition is moderate in one phase and extreme in the other with the asymptotic temperature ratio diverging. Sufficiently light impurities are governed by the fluid, and their scaled velocity distribution has similar extremal statistics as has the fluid. In one dimension, the scaled impurity velocity distribution is given by a series containing powers of Lorentzians of all orders (the infinite sum truncates for a few special values of the impurity restitution coefficient). Interestingly there is a series of increasing transition masses that corresponds to divergence of the ratio of velocity moments of decreasing order. These masses are always larger than unity and the largest such mass corresponds to the temperature. When the impurity mass is larger than this critical mass, the impurity and the fluid effectively decouple. Next, we considered collision rates that account for the temperature difference between the fluid and the impurity. In this case, the primary transition corresponding to the temperature is suppressed but further transitions corresponding to divergence of higher moments may still occur.

Many more quantities should be tractable in the framework of the Maxwell model. For example, one can study velocity correlations as well as the velocity autocorrelation functions. Furthermore, corrections to the leading asymptotic behavior, especially in the heavy impurity phase, can be systematically evaluated. Additionally, this model can be applied to particles of different sizes by modifying the collision cross sections.

We reiterate that the inelastic Maxwell model exactly describes only a mean-field collision process where the collision partners are random and the impact directions are chosen according to a uniform distribution. On the other hand, it is an uncontrolled approximation when applied to real granular fluids. At best, this model may be used to approximate sufficiently low velocity moments. Still, these results show that the relative collision rates play an important role. Comparison with the hard sphere Boltzmann equation, Molecular Dynamics simulations, and experiment is needed to gauge the utility of either of the two approximate models we treated. In practice, the impurity problem involves an additional level of difficulty. Gathering meaningful statistics requires many replica systems with one impurity or alternatively, the vanishing volume fraction limit of a mixture.

This research was supported by DOE (W-7405-ENG-36), NSF(DMR9978902), and ARO (DAAD19-99-1-0173).

---

[1] J. C. Williams, Powder Technology **15**, 245 (1976).
[2] C. S. Campbell, Annu. Rev. Fluid Mech. **22**, 57 (1990).
[3] A. Rosato, K. J. Strandburg, F. Prinz, and R. H. Swendsen, Phys. Rev. Lett. **58**, 1038 (1987).
[4] F. J. Alexander and J. L. Lebowitz, J. Phys. A **23**, L375 (1990).
[5] R. Jullien, P. Meakin, and A. Pavlovitch, Phys. Rev. Lett. **69**, 640 (1992).
[6] J. Duran, J. Rajchenbach, and E. Clement, Phys. Rev. Lett. **70**, 2431 (1993).
[7] J. B. Knight, H. M. Jaeger, and S. R. Nagel, Phys. Rev. Lett. **70**, 3728 (1993).
[8] J. Duran and R. Jullien, Phys. Rev. Lett. **80**, 3547 (1998).
[9] D. C. Hong, P. V. Quinn, and S. Luding, Phys. Rev. Lett. **86**, 3423 (2001).
[10] L. Trujillo and H. J. Herrmann, *cond-mat*/0202484.
[11] R. D. Wildman and D. J. Parker, Phys. Rev. Lett. **88**, 064301 (2002).
[12] A. Santos and J. W. Dufty, Phys. Rev. Lett. **86**, 4823 (2001); Phys. Rev. E **64**, 051305 (2001).
[13] J. C. Maxwell, Phil. Trans. R. Soc. **157**, 49 (1867).
[14] C. Truesdell and R. G. Muncaster, *Fundamentals of Maxwell's Kinetic Theory of a Simple Monoatomic Gas* (Academic Press, New York, 1980).
[15] M. H. Ernst, Phys. Reports **78**, 1 (1981).
[16] A. V. Bobylev, Sov. Sci. Rev. C. Math. Phys. **7**, 111 (1988).
[17] E. Ben-Naim and P. L. Krapivsky, Phys. Rev. E **61**, R5 (2000).
[18] A. V. Bobylev, J. A. Carrillo, and I. M. Gamba, J. Stat. Phys. **98**, 743 (2000).
[19] A. V. Bobylev and C. Cercignani, J. Stat. Phys. **106**, 547 (2002).
[20] A. Baldassarri, U. M. B. Marconi, and A. Puglisi, Europhys. Lett. **58**, 14 (2002).
[21] P. L. Krapivsky and E. Ben-Naim, J. Phys. A **35**, L147 (2002); E. Ben-Naim and P. L. Krapivsky, *cond-mat*/0202332.
[22] M. H. Ernst and R. Brito, Europhys. Lett. **58**, 182 (2002), and *cond-mat*/0112417.
[23] U. M. B. Marconi and A. Puglisi, *cond-mat*/0112336 and *cond-mat*/0202267.
[24] A. Santos, *cond-mat*/0204071.
[25] P. K. Haff, J. Fluid Mech. **134**, 401 (1983).
[26] I. Goldhirsch and G. Zanetti, Phys. Rev. Lett. **73** 1619 (1993).
[27] F. B. Pidduck, Proc. London Math. Soc. Ser. *2*, **15**, 89 (1915).
[28] P. A. Martin and J. Piasecki, Europhys. Lett. **46**, 613 (1999).
[29] H. A. Lorentz, Arch. Néerl. **10**, 336 (1905); for a review, see E. H. Hauge, in: *Transport Phenomena*, eds. G. Kirczenow and J. Marro (Springer-Verlag, Berlin, 1974).
[30] C. S. Campbell and C. E. Brenner, J. Fluid Mech. **151**, 167 (1985).
[31] S. McNamara and W. R. Young, Phys. Rev. E **53**, 5089 (1996).
[32] T. P. C. van Noije and M. H. Ernst, in *Granular Gases*, edited by T. Pöschel and S. Luding (Springer, Berlin, 2000), p. 3; I. Goldhirsch, *ibid*, p. 79.
[33] J. W. Dufty, *cond-mat*/0108444.