Maxwell Model of Traffic Flows

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We investigate traffic flows using the kinetic Boltzmann equations with a Maxwell collision integral. This approach allows analytical determination of the transient behavior and the size distributions. The relaxation of the car and cluster velocity distributions towards steady state is characterized by a wide range of velocity dependent relaxation scales, $R^{1/2} < \tau(v) < R$, with R the ratio of the passing and the collision rates. Furthermore, these relaxation time scales decrease with the velocity, with the smallest scale corresponding to the decay of the overall density. The steady state cluster size distribution follows an unusual scaling form $P_m \sim \langle m\rangle^{-4}\Psi(m/\langle m\rangle^2)$. This distribution is primarily algebraic, $P_m \sim m^{-3/2}$, for $m \ll \langle m\rangle^2$, and is exponential otherwise.

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I. INTRODUCTION

Traffic flows exhibit a variety of collective behaviors typical to non-equilibrium systems [1–5]. The observed phenomenology is rich and includes shock waves, traffic jams, clustering, and synchronized flow [4–7]. A number of models and theoretical approaches including fluid mechanics [2,6,8], cellular automata [9–17], particle hopping [18–20], ballistic motion [21–28], and optimal velocity [29–31] are used to describe these observations. Yet, different approaches have different virtues, e.g., kinetic theory is more appropriate for dilute flows, while fluid mechanics is more appropriate for dense flows.

Here, we focus on the kinetic description of traffic. Previously, we introduced a microscopic ballistic motion model and used it to derive Boltzmann Equations (BE) for traffic on a one lane roadway [24]. A generalization to situations where passing is allowed shows that a transition from a low-density "laminar" flow to a high-density "congested" flow generally occurs [25,26]. The resulting BE are technically difficult, and a number of important questions remain unresolved including the transient characteristics and the cluster-size distribution. Indeed, previous studies addressed only steady state properties and the results concerned mainly the velocity distributions.

Our goal is to obtain these relevant flow characteristics. To this end, we propose an approach inspired by Maxwell's classical model, widely used in kinetic theory [32,33]. This Maxwell approach uses a velocity independent collision rate, thereby considerably simplifying the analysis. In fact, upon transforming the kinetic equations from integral into differential ones, the Maxwell model results in *first* order differential equations while the Boltzmann approach leads to *second* order equations.

We will show that the Maxwell approximation is faithful to the nature of the original traffic equations as it qualitatively reproduces transient characteristics for no passing zones, as well as steady state characteristics for passing zones. We further find that the cluster velocity distribution approaches its steady state according to a wide spectrum of relaxation scales, with the smallest describing decay of the overall cluster density. Furthermore, the size distribution is characterized by a strong algebraic tail for small and average sizes, while it is exponentially small for large size.

II. THE MAXWELL MODEL

The ballistic motion approach models the basic processes underlying one lane traffic flows: passing and slowing down due to clustering. The main assumption is that each driver has a fixed intrinsic velocity. The driving rules are as follows: A car moves with constant intrinsic velocity on a one lane road until it overtakes a smaller velocity car or a cluster. After such an encounter, or "collision", the incident car immediately adopts this smaller velocity, thereby joining a cluster. Cars may also resume driving with their intrinsic velocities, and such passing events are assumed to occur with a constant rate. This model is an idealized description of one lane traffic flows as several time and length scales including the actual collision time, the passing time, and the car size are neglected.

Let $P(v,t)$ be the density of clusters moving with velocity v at time t, and let $P_0(v)$ be the intrinsic velocity distribution. Natural initial conditions where cars are randomly distributed in space and drive with their intrinsic velocities are imposed, i.e., $P(v,0) = P_0(v)$. Under the assumption that space and velocity remain uncorrelated, a mean-field Boltzmann equation is written

$$\frac{\partial P(v,t)}{\partial t} = t_0^{-1}\left[P_0(v) - P(v,t)\right] - P(v,t)\int_0^v dv' U(v,v')P(v',t). \tag{1}$$

The first term on the right-hand side represents cars escaping their respective clusters with a constant rate t_0^{-1}. The last term accounts for decrease in the cluster density due to collisions. For traffic flows the collision rate should read $U(v,v') = v - v'$. For such a collision rate, steady state properties of the velocity distributions can be obtained by transforming the rate equation into a second order nonlinear differential equation [25]. However, a number of important characteristics including the size distribution and time dependent properties appear to be analytically intractable.

We propose using a constant collision rate, $U(v, v') = u_0$ to simplify the above traffic equations. Similar in spirit approximations, termed the Maxwell Model (MM), proved useful in kinetic theory [32,33]. A natural choice for the constant rate u_0 is the average velocity difference, $u_0 = \langle v - v' \rangle \propto \langle v \rangle$. One may wonder whether such an approximation is reasonable for traffic flows. Ignoring $P(v', t)$ in the collision integral $I(v) = \int_0^v dv' U(v, v') P(v', t)$, we have $I(v) \propto v$ for the MM, while $I(v) \propto v^2$ for the BE. Hence, the integral remains an increasing function of the velocity. Furthermore, cars must slow down before a collision, and therefore, the collision rate should be slower than linear. The MM can actually be considered as the limiting case of zero deceleration, while the linear rate corresponds to the limit of infinite deceleration.

Let c_0 be the initial car concentration, v_0 the average intrinsic velocity, t_0^{-1} the passing rate, and u_0 the collision rate. Introducing the dimensionless velocity $v/v_0 \to v$, space $xc_0 \to x$, and time $c_0 v_0 t \to t$ variables normalizes the initial concentration and typical velocity to unity. The master equation (1) is characterized by two dimensionless numbers

$$\frac{1}{\nu} \frac{\partial P(v,t)}{\partial t} = \frac{1}{R} [P_0(v) - P(v,t)] - P(v,t) \int_0^v dv' P(v', t). \quad (2)$$

The normalized collision rate, $\nu = u_0 / v_0$, merely rescales time. Thus, it is set to unity without loss of generality. The number $R = c_0 u_0 t_0 = t_{\rm esc}/t_{\rm col}$, equals the ratio of the two elementary time scales, the escape time $t_{\rm esc} = t_0$, and the collision time $t_{\rm col} = (c_0 u_0)^{-1}$. This number, termed the "collision number", plays an important role in determining the nature of traffic flows.

We will show below that the Maxwell model is completely solvable. Although quantitative results of the MM may differ from the BE, they faithfully reproduce the qualitative behavior of the traffic equations.

III. THE CLUSTER VELOCITY DISTRIBUTION

We start with steady-state and time dependent properties of the cluster velocity distribution. Let us introduce the auxiliary function

$$Q(v,t) = R^{-1} + \int_0^v dv' P(v', t), \quad (3)$$

which gives the cluster distribution via differentiation $P(v,t) = \partial Q(v,t) / \partial v$. This auxiliary function enables us to reduce the integro-differential Eq. (2) into the differential equation

$$\frac{\partial}{\partial t} \frac{\partial Q}{\partial v} = \frac{1}{R} \frac{\partial Q_0}{\partial v} - Q \frac{\partial Q}{\partial v}. \quad (4)$$

This equation can be integrated over v, and using the boundary condition $Q|_{v=0} = R^{-1}$ we find

$$\frac{\partial Q(v,t)}{\partial t} = \frac{Q_0(v)}{R} - \frac{Q^2(v,t)}{2} - \frac{1}{2R^2}. \quad (5)$$

Integrating Eq. (5), the auxiliary function is obtained explicitly for arbitrary initial conditions

$$Q(v,t) = Q_\infty(v) \frac{1 + A(v) e^{-tQ_\infty(v)}}{1 - A(v) e^{-tQ_\infty(v)}}, \quad (6)$$

with notation $A(v) = [Q_0(v) - Q_\infty(v)]/[Q_0(v) + Q_\infty(v)]$. Here we use the subscript ∞ to denote steady state. The steady state auxiliary function $Q_\infty(v) \equiv Q(v, t = \infty)$ is given by

$$Q_\infty(v) = R^{-1} \left[1 + 2R \int_0^v dv' P_0(v') \right]^{1/2}. \quad (7)$$

Since the concentration is obtained from $Q(v,t)$ using $c(t) = \lim_{v \to \infty} [Q(v,t) - R^{-1}]$, and since the cluster velocity distribution is obtained by differentiation, Eq. (6) represents a complete explicit solution of the Maxwell model.

We first examine steady state properties of the cluster velocity distribution. Comparing with the corresponding behavior emerging from the BE will allow us to test the utility of the Maxwell model. Evaluating the infinite velocity limit of the auxiliary function gives the overall cluster density

$$c_\infty = R^{-1} \left(\sqrt{1 + 2R} - 1 \right). \quad (8)$$

A remarkable feature of the steady state cluster density is that it is a function of the collision number only. Such independence of the initial velocity distribution has been observed in a few other ballistic aggregation problems [29,34]. Note that $c_\infty = 1 - R/2 + \mathcal{O}(R^2)$ for $R \ll 1$, i.e., the difference from the initial density is of order R in the laminar flow regime. In this study, we will focus on the complementary nontrivial limit of congested flows, i.e., $R \gg 1$. Here, the cluster concentration is significantly reduced, $c_\infty \sim R^{-1/2}$, and large clusters with an average size $\langle m \rangle = c_\infty^{-1} \sim R^{1/2}$ form in agreement with the BE results.

The cluster velocity distribution is obtained from Eq. (7) by differentiation

$$P_\infty(v) = P_0(v) \left[1 + 2R \int_0^v dv' P_0(v') \right]^{-1/2}. \quad (9)$$

When $R \ll 1$, the difference between the initial and the steady state distributions is of order R. This indicates a laminar flow regime when the correction due to collisions is small and can be obtained by expanding the solution perturbatively around the initial state. When $R \gg 1$, we use the notation $I_0(v) = \int_0^v dv' P_0(v')$ and write the leading behavior of Eq. (9) as

$$P_\infty(v) \simeq \begin{cases} P_0(v) & v \ll v^*; \\ P_0(v)[2RI_0(v)]^{-1/2} & v \gg v^*. \end{cases} \tag{10}$$

The two limiting behaviors match at the threshold velocity v^* which is found from $1 \sim RI_0(v^*) = R\int_0^{v^*} dv\, P_0(v)$. In agreement with the Boltzmann approach [25,26], a boundary layer structure is found for the velocity distribution, where in the inner region the original distribution prevails, while in the outer region, the distribution is substantially reduced. The average cluster velocity remains of order unity. Additionally, the suppression of the fastest velocities is proportional to the concentration, again in agreement with the BE results. We conclude that although the MM differs quantitatively from the exact BE behavior, it qualitatively reproduces the steady state behavior.

We turn now to analyzing the transient properties and in particular the approach towards steady state. The time dependent concentration reads

$$c(t) = c_\infty \frac{1 + Be^{-t/\tau_c}}{1 - Be^{-t/\tau_c}}, \tag{11}$$

with the constant $B = A(\infty) = (1 - c_\infty)/(1 + c_\infty)$ and the relaxation time $\tau_c = R/\sqrt{2R+1}$ corresponding to the concentration decay. We see that the cluster concentration exponentially approaches its steady state value

$$c(t) \simeq c_\infty \left(1 + 2Be^{-t/\tau_c}\right). \tag{12}$$

As the distribution changes slightly in the laminar phase, relaxation times remain of order unity when $R \ll 1$. However, for congested flows the relaxation time diverges with the collision number $\tau_c \sim R^{1/2}$.

The explicit time dependent auxiliary function allows determination of relaxation properties of the cluster velocity distribution. In the long time limit Eq. (6) reads

$$Q(v,t) = Q_\infty(v)\left[1 + 2A(v)e^{-t/\tau(v)}\right] \tag{13}$$

with the velocity dependent relaxation time $\tau(v) = 1/Q_\infty(v)$. Thus, the steady state properties are reflected in the transient characteristics. The velocity dependence of the relaxation time $\tau(v)$ becomes especially pronounced for large collision numbers where it exhibits the following boundary layer structure

$$\tau(v) \sim \begin{cases} R & v \ll v^*; \\ [R/I_0(v)]^{1/2} & v \gg v^*. \end{cases} \tag{14}$$

For sufficiently small velocities, the collision integral is negligible, and the relaxation time R, suggested by Eq. (2) holds. While small velocities are governed by (almost) velocity independent relaxation scales, large velocities are characterized by velocity dependent decay rates. Furthermore, a large range of relaxation scales exists $R^{1/2} < \tau < R$ with the larger relaxation scales corresponding to smaller velocities. This is consistent with dimensional arguments that time and velocity are inversely related. Interestingly, the smallest possible relaxation scale corresponds to the overall cluster density.

One anticipates that the relaxation time $\tau(v)$ also governs the decay of the cluster velocity distribution $P(v,t)$. This is indeed the case. To obtain explicit expressions we first simplify Eq. (13),

$$Q(v,t) - Q_\infty(v) \simeq \begin{cases} RI_0^2(v)e^{-t/\tau(v)} & v \ll v^*; \\ 2Q_\infty(v)e^{-t/\tau(v)} & v \gg v^*. \end{cases} \tag{15}$$

Differentiating with respect to v gives the cluster velocity

$$P(v,t) - P_\infty \simeq \begin{cases} 2RP_0(v)I_0(v)e^{-t/\tau(v)} & v \ll v^{**}; \\ -RP_0(v)I_0^2(v)[te^{-t/\tau(v)}] & v^{**} \ll v \ll v^*; \\ -2R^{-1}P_0(v)[te^{-t/\tau(v)}] & v \gg v^*; \end{cases}$$

with $\tau(v)$ given by Eq. (14). The expressions match at the boundary velocities which are determined from $RI_0(v^*) \sim 1$ and $tI_0(v^{**}) \sim 1$. Only for velocities slower the decaying boundary velocity $v^{**}(t)$ is the correction to the cluster density positive. This is surprising since both the overall cluster density and the auxiliary function exhibit positive corrections, as one would naively expect since $P_0(v) > P_\infty(v)$.

Since the relaxation times diverge with increasing R, an intermediate behavior should emerge in the time range $t \ll \sqrt{R}$. In this regime, the system has not yet "realized" that passing is allowed, and the behavior should agree with the no passing case where $R = \infty$. By directly solving Eq. (5) with $R^{-1} = 0$ one finds

$$P(v,t) = \frac{P_0(v)}{\left[1 + \frac{1}{2}tI_0(v)\right]^2}. \tag{16}$$

For arbitrary intrinsic velocity distribution, a scaling asymptotic behavior emerges

$$P(v,t) \simeq \frac{c(t)}{\langle v(t)\rangle} F\left(\frac{v}{\langle v(t)\rangle}\right), \tag{17}$$

with the cluster concentration $c(t) \sim t^{-1}$ and the average velocity determined by $tI_0\left(\langle v(t)\rangle\right) \sim 1$. We see that the average velocity in the no passing case is proportional to the time dependent boundary velocity: $\langle v(t)\rangle \sim v^{**}$. When the leading small velocity behavior of the intrinsic velocity distribution is algebraic, $P_0(v) \sim v^\mu$ as $v \to 0$, the average velocity decays as a power law in time, $\langle v(t)\rangle \sim t^{-\beta}$ with $\beta = 1/(\mu+1)$. Comparing with the exact behavior in the no passing limit of ballistic motion model, we see that the overall scaling picture is reproduced, while the quantitative details and in particular the scaling exponents are different.

To summarize, explicit expressions for all cluster properties are possible in the realm of the MM. The relaxation towards steady state occurs in two stages. The early one, $t \ll \sqrt{R}$, corresponds to a no passing intermediate asymptotics. Then, the passing mechanism comes into play, and the system approaches steady state. This latter relaxation is nontrivial in several ways. The decay is non-uniform in time as a wide range of time scales are observed. It is also non-uniform in velocity as the cluster velocity distribution involves three layers of greatly different width, i.e. it exhibits the so-called "triple deck" structure [35]. The first layer $v \ll v^{**}(t)$ (referred to as the lower deck) shrinks with time and the velocity distribution in this deck approaches the steady state exponentially from above. In the middle and upper decks, the approach towards steady state is from below and has a linear in time correction to the exponential decay.

IV. THE CAR VELOCITY DISTRIBUTION

The cluster velocity distribution does not provide the observed distribution of car velocities since all clusters – large and small – are taken with equal weight. In what follows, we concentrate on the car velocity distribution, which determines basic properties such as the flux.

Within the framework of the MM, the car velocity distribution satisfies

$$\frac{\partial G(v,t)}{\partial t} = R^{-1}[P_0(v) - G(v,t)] \qquad (18)$$
$$-G(v,t)\int_0^v dv' P(v',t) + P(v,t)\int_v^\infty dv' G(v',t).$$

The escape term is the sum of a gain term $R^{-1}[P_0 - P]$ and a loss term $-R^{-1}[G-P]$. In a collision between two clusters, all cars belonging to the faster cluster acquire the slower cluster velocity. Thus, in both collision terms the argument of P is smaller than the argument of G. In contrast with Eq. (2) collisions can now lead to a gain in the car velocity distribution. Although the integration limits resemble those of the previous kinetic equations [23], the collision terms are different, a reflection of the different treatment of cars and clusters in this theory. One can verify that Eq. (18) conserves the car density $1 = \int_0^\infty dv\, G(v,t)$. An alternative approach for obtaining $G(v,t)$ involves the conditional velocity distribution $P(v,v',t)$. This more detailed distribution can also be used to verify $G(v,t)$, and for completeness, we detail its derivation in Appendix A.

Let us introduce the auxiliary function

$$g(v,t) = \int_v^\infty dv'\, G(v',t). \qquad (19)$$

In terms of the auxiliary functions g, Q, and Q_0, Eq. (18) becomes

$$\frac{\partial}{\partial t}\frac{\partial g(v,t)}{\partial v} = -\frac{\partial}{\partial v}\left[\frac{Q_0(v)}{R} + g(v,t)Q(v,t)\right]. \qquad (20)$$

Integrating over the velocity and using $g_0(v) = Q_0(\infty) - Q_0(v)$ gives the master equation

$$\frac{\partial g(v,t)}{\partial t} = R^{-1}g_0(v) - g(v,t)Q(v,t). \qquad (21)$$

We first analyze the steady state properties which are obtained immediately from Eq. (21)

$$g_\infty(v) = \frac{g_0(v)}{RQ_\infty(v)}. \qquad (22)$$

Interestingly, this auxiliary function and the cluster velocity distribution experience the same relative reduction at the steady state, $g_\infty(v)/g_0(v) = P_\infty(v)/P_0(v) = 1/RQ_\infty(v)$. Differentiating $g_\infty(v)$, we get

$$G_\infty(v) = P_0(v)\,\frac{1+R+RI_0(v)}{[1+2RI_0(v)]^{3/2}}. \qquad (23)$$

In the congested phase, $R \gg 1$, the car velocity distribution has the following limiting behaviors:

$$G_\infty(v) \sim \begin{cases} RP_0(v), & v \ll v^*; \\ R^{-1/2}P_0(v)I_0^{-3/2}(v), & v \gg v^*. \end{cases} \qquad (24)$$

Thus, while the fast tail decay $R^{-1/2}$ agrees with the Boltzmann equation approach [25], the slow tail enhancement R is larger than the Boltzmann result where this enhancement is of the order R^α with $0 \le \alpha \le 1$.

The car velocity distribution immediately gives the average size of a v-cluster

$$\langle m(v)\rangle = \frac{1+R+RI_0(v)}{1+2RI_0(v)}, \qquad (25)$$

obtained from $\langle m(v)\rangle = G(v)/P(v)$. As expected, the average cluster size decreases with the velocity. The average cluster size obeys the bounds $1 \le \langle m(v)\rangle \le 1+R$, with the upper (lower) bound achieved by the slowest (fastest) clusters. An additional quantity, immediately derived from the car velocity distribution is the flux, $J_\infty = \int dv\, v\, G_\infty(v)$. One can use Eq. (23) to find

$$J_\infty = \int_0^\infty dv\, \frac{1-I_0(v)}{\sqrt{1+2RI_0(v)}}. \qquad (26)$$

In the congested phase, the flux is proportional to the threshold velocity, $J_\infty \sim v^*$, in agreement with the Boltzmann equation results.

We now focus on the time dependent behavior. Integration of equation (21) gives an explicit expression for $g(v,t)$ (for a derivation, see Appendix B)

$$\frac{g(v,t)}{g_\infty(v)} = \frac{Q(v,t)}{Q_\infty(v)} + \frac{Q^2(v,t)-Q_\infty^2(v)}{Q_\infty(v)}\left[\frac{1}{I_0(v)} - \frac{t}{2}\right] \qquad (27)$$

The relaxation of g follows directly from the relaxation of Q since $g(v,t) - g_\infty(v) \propto Q(v,t) - Q_\infty(v)$ when $t \to \infty$.

Using Eq. (15), we evaluate the leading relaxation behavior of $g(v,t)$

$$g(v,t)-g_\infty \simeq \begin{cases} 2Rg_0(v)I_0(v)e^{-t/\tau(v)} & v \ll v^{**}; \\ -Rg_0(v)I_0^2(v)[te^{-t/\tau(v)}] & v^{**} \ll v \ll v^*; \\ -2R^{-1}g_0(v)[te^{-t/\tau(v)}] & v \gg v^*. \end{cases}$$

Interestingly, the relaxation of the (properly normalized) cluster and auxiliary car velocity distribution are identical, $[g(v,t)-g_\infty]/g_0(v) \simeq [P(v,t)-P_\infty]/P_0(v)$. Relaxation of the car velocity distribution is obtained from $G = -\partial g/\partial v$

$$\frac{G(v,t)}{G_\infty(v)} - 1 \simeq \begin{cases} -2e^{-t/\tau(v)} & v \ll v^{**}; \\ -I_0^2(v)[t^2e^{-t/\tau(v)}] & v^{**} \ll v \ll v^*; \\ -I_0(v)R^{-1}[t^2e^{-t/\tau(v)}] & v \gg v^*. \end{cases}$$

Thus an exponential relaxation with a velocity dependent time scale $\tau(v)$ underlies the approach of all velocity distributions towards steady state. The car velocity distribution exhibits the triple deck structure similar to that of the cluster velocity distribution. Some details change however; for example, in the middle and upper decks the prefactor t^2 further slows down the decay of $G(v,t)$. The car velocity distribution approaches its steady state always from below.

V. THE SIZE DISTRIBUTION

An important characteristic of traffic flows, the cluster size distribution, has been determined analytically only in the no passing limit [24]. We now address this issue in the framework of the MM. Let us consider $P_m(t)$ the cluster size distribution which evolves according to the following rate equation

$$\frac{\partial P_m(t)}{\partial t} = R^{-1}[mP_{m+1}(v,t) - (m-1)P_m(t)] \tag{28}$$
$$+R^{-1}\delta_{m,1}[1-c(t)] - c(t)P_m(t) + \frac{1}{2}\sum_{i+j=m} P_i(t)P_j(t).$$

Terms proportional to R^{-1} account for escape, while the rest represent collisions. The overall collision rate experienced by a cluster, $c(t)$, is velocity-independent. These rate equations are reminiscent of an aggregation-fragmentation process [36,37]. Indeed, collisions lead to cluster aggregation while passing events split clusters.

Since aggregation and fragmentation are opposite mechanisms, their combined effect generally leads to a steady state. We leave the ambitious task of a complete solution for the future, and restrict our attention to the steady state where Eqs. (28) read

$$c_\infty P_m = R^{-1}[mP_{m+1} - (m-1)P_m] + R^{-1}\delta_{m,1}(1-c_\infty) + \frac{1}{2}\sum_{i+j=m} P_iP_j. \tag{29}$$

It is useful to introduce the generating function

$$\mathcal{F}(z) = c_\infty^{-1}\sum_m z^m P_m. \tag{30}$$

At the steady state, it satisfies the Riccati equation

$$\mathcal{F}^2 - 2\mathcal{F} + z + \frac{c_\infty}{1-c_\infty} z(1-z)\frac{d}{dz}\left(\frac{\mathcal{F}}{z}\right) = 0. \tag{31}$$

The identity $(1-c_\infty)/(Rc_\infty^2) = 1$ was used in obtaining this equation.

Although we could not solve these equations generally, most of the interesting features can be obtained by carefully analyzing the leading terms in R. The asymptotic relation $c_\infty \simeq \sqrt{2/R}$ suggests that the last term in Eq. (31) is negligible. Solving the resulting quadratic equation subject to the boundary condition $\mathcal{F}(1) = 1$ gives $\mathcal{F} = 1 - \sqrt{1-z}$. Expanding this expression in powers of z we arrive at

$$P_m = c_\infty \frac{\Gamma\left(m-\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{2}\right)\Gamma(m+1)}, \tag{32}$$

which simplifies to $P_m \simeq (2\pi R)^{-1/2}m^{-3/2}$ for $m \gg 1$. However, this solution does not apply for very large m, or equivalently near $z = 1$. This follows e.g. from the conservation of the car density, $\sum_m mP_m = 1$, which implies that a crossover from (32) to the tail behavior should occur at the cutoff size $m_c \sim \langle m\rangle^2 \sim R$.

To investigate the very large m behavior we have to return to Eq. (31). Fortunately, in the proximity of $z = 1$, i.e., when $1 - z \sim R^{-1}$, the generating function depends on a single scaling variable

$$1 - \mathcal{F} = c_\infty\Phi(\zeta), \quad \text{with} \quad \zeta = \frac{1-z}{c_\infty^2}. \tag{33}$$

This can be seen by balancing the leading terms in Eq. (31). The scaling function Φ satisfies the Riccati equation

$$\zeta\,\Phi'(\zeta) = \zeta - \Phi^2 \tag{34}$$

subject to the boundary condition $\Phi(0) = 0$. Using the transformation $\Phi(\zeta) = \phi(\zeta)/\phi'(\zeta)$ reduces Eq. (34) to a second order linear differential equation

$$\zeta\,\phi''(\zeta) = \phi(\zeta). \tag{35}$$

This is a solvable one-dimensional Shrödinger equation for a particle with zero energy in a repulsive Coulomb potential. Indeed, a solution is found by reducing Eq. (35) to the Bessel equation. Choosing the solution which satisfies the appropriate boundary conditions, $\phi = 0, \phi'(\zeta) = 1$ at $\zeta = 0$, one finds $\phi(\zeta) = \sqrt{\zeta}\,I_1(2\sqrt{\zeta})$ with $I_1(x)$ the modified Bessel function of order one. Returning to $\Phi(\zeta)$ we obtain

$$\Phi(\zeta) = 2\zeta\left[1 + 2\sqrt{\zeta}\,\frac{I_1'(2\sqrt{\zeta})}{I_1(2\sqrt{\zeta})}\right]^{-1} = \sqrt{\zeta}\,\frac{I_1(2\sqrt{\zeta})}{I_0(2\sqrt{\zeta})}. \quad (36)$$

The last expression is derived using the identities $I_1'(x) = I_0(x) - x^{-1}I_1(x)$ and $I_0'(x) = I_1(x)$ [38].

The function $\Phi(\zeta)$ is the Laplace transform of the properly scaled size distribution. Indeed, Eq. (33) implies $\sum P_m(1-z^m) = c_\infty^2\Phi[c_\infty^{-2}(1-z)]$ whose inversion yields the scaling form $P_m(R) = c_\infty^4\Psi(c_\infty^2 m)$. Therefore, in the large R limit the size distribution follows the scaling form

$$P_m \simeq \frac{1}{\langle m\rangle^4}\Psi\left(\frac{m}{\langle m\rangle^2}\right), \quad (37)$$

with $\langle m\rangle = 1/c_\infty \simeq \sqrt{R/2}$. The scaling function $\Psi(M)$ obeys $\Phi(\zeta) = \int_0^\infty dM\,\Psi(M)\left[1 - e^{-\zeta M}\right]$. Differentiating both sides with respect to ζ shows that $\Phi'(\zeta)$ is simply the Laplace transform of $M\Psi(M)$

$$\Phi'(\zeta) = \int_0^\infty dM\,M\Psi(M)\,e^{-\zeta M}. \quad (38)$$

Consequently, the asymptotic behavior of the size distribution can be determined from the corresponding asymptotics of $\Phi(\zeta)$. The latter are found from Eq. (36):

$$\Phi(\zeta) \simeq \begin{cases} \zeta^*(\zeta+\zeta^*)^{-1} & \zeta \to -\zeta^*; \\ \sqrt{\zeta} & \zeta \to \infty. \end{cases} \quad (39)$$

The algebraic behavior of $\Phi(\zeta)$ at large ζ implies an algebraic behavior of $\Psi(M)$ at small M; similarly, the pole at $\zeta = -\zeta^*$ ($\zeta^* \cong 1.445796$ [39]) implies an exponential decay for large M:

$$\Psi(M) \simeq \begin{cases} (4\pi)^{-1/2}M^{-3/2}, & M \ll 1; \\ \zeta^*\exp(-\zeta^* M), & M \gg 1. \end{cases} \quad (40)$$

In terms of the original variables we have

$$P_m \simeq \begin{cases} (2\pi R)^{-1/2}m^{-3/2}, & m \ll R; \\ 4\zeta^* R^{-2}\exp(-2\zeta^* m/R), & m \gg R. \end{cases} \quad (41)$$

These two limiting behaviors match at $m \sim R$ where $P_m \sim R^{-2}$. Additionally, the value of the cutoff size, $m_c \sim R$, agree with our previous findings.

In conclusion, the Maxwell equation approach allows explicit calculations of the size distribution. It decays algebraically with size for small and average clusters, and exponentially for very large clusters. The interesting aspect of the size distribution concerns its scaling form. If the typical and the average size would be the same, a naive scaling argument $m/\langle m\rangle$ would underly the size distribution. However, a different picture emerges where the scaling variable is $m/\langle m\rangle^2$. Indeed, Eq. (41) is consistent with a typical size of order unity, in contrast with the growing average size $\langle m\rangle \sim \sqrt{R}$, a reflection of the anomalous algebraic behavior of the size distribution below the cutoff size.

VI. THE SIZE-VELOCITY DISTRIBUTION

So far, we have addressed velocity and size distributions separately. However, size and velocity are coupled in a nontrivial manner, and for example, slower clusters should be larger than faster ones. We thus consider $P_m(v,t)$, the distribution of clusters of size m and velocity v. This joint distribution evolves according to

$$\begin{aligned}\frac{\partial P_m(v,t)}{\partial t} &= R^{-1}[mP_{m+1}(v,t) - (m-1)P_m(v,t)] \\ &+ R^{-1}\delta_{m,1}[P_0(v) - P(v,t)] - c(t)P_m(v,t) \\ &+ \int_v^\infty dv' \sum_{i+j=m} P_i(v',t)P_j(v,t).\end{aligned} \quad (42)$$

The car and cluster velocity distributions are simply the zeroth and first moment of the size distribution, $P(v,t) = M_0(v,t)$ and $G(v,t) = M_1(v,t)$, with $M_a(v,t) = \sum_m m^a P_m(v,t)$. Consequently, the respective evolution equations are recovered by summation of Eqs. (42) over m. Furthermore, integration over the velocities gives the size distribution and Eq. (28) is recovered by using $P_m(t) = \sum_m P_m(v,t)$.

It proves useful to introduce the auxiliary functions $Q_m(v,t) = \int_v^\infty dv' P_m(v',t)$. The cluster size distribution can be expressed through these auxiliary functions, $P_m(t) = Q_m(0,t)$. Additionally, the identity $Q(v,t) + \sum Q_m(v,t) = R^{-1} + c(t)$ holds. Integrating Eqs. (42) over v gives

$$\begin{aligned}\frac{\partial Q_m(v,t)}{\partial t} &= R^{-1}[mQ_{m+1}(v,t) - (m-1)Q_m(v,t)] \\ &+ R^{-1}\delta_{m,1}q(v,t) - c(t)Q_m(v,t) \\ &+ \frac{1}{2}\sum_{i+j=m} Q_i(v,t)Q_j(v,t)\end{aligned} \quad (43)$$

with $q(v,t) = \int_v^\infty dv'[P_0(v',t) - P(v',t)]$ or alternatively $q(v,t) = 1 - c(t) + Q(v,t) - Q_0(v)$. In deriving (43) we used the following boundary conditions: $Q_m = 0$, $Q_0 = 1 + R^{-1}$, and $Q = c(t) + R^{-1}$ at $v = \infty$. Since the velocity plays the role of a parameter, Eqs. (43) can be treated as ordinary differential equations. We again restrict our attention to the steady state where

$$\begin{aligned}c_\infty Q_m(v) &= R^{-1}[mQ_{m+1}(v) - (m-1)Q_m(v)] \\ &+ R^{-1}\delta_{m,1}q(v) + \frac{1}{2}\sum_{i+j=m} Q_i(v)Q_j(v),\end{aligned} \quad (44)$$

with $q(v) = q_\infty(v) = 1 - c_\infty + Q_\infty(v) - Q_0(v)$. Introducing the generating function

$$\mathcal{Q}(z,v) = c_\infty^{-1}\sum_{m=1}^\infty z^m Q_m(v) \quad (45)$$

reduces Eqs. (44) into a set (parameterized by v) of Riccati equations for $\mathcal{Q} = \mathcal{Q}(z,v)$:

$$\mathcal{Q}^2-2\mathcal{Q}+z\frac{q(v)}{q(0)}+\frac{c_\infty}{1-c_\infty}z(1-z)\frac{\partial}{\partial z}\left(\frac{\mathcal{Q}}{z}\right)=0 \quad (46)$$

This Riccati equation reduces to Eq. (31) when $v=0$. The above treatment of the size distribution suggests that the derivative term in Eq. (46) is negligible for sufficiently small sizes. In this case, Eq. (46) simplifies to $\mathcal{Q}^2-2\mathcal{Q}+zq(v)/q(0)=0$ which is solved to give $\mathcal{Q}(z,v)=1-\sqrt{1-zq(v)/q(0)}$. Using the large R behavior $q(v)/q(0)\to 1+Q(v)-Q_0(v)$ yields

$$\mathcal{Q}(z,v)\simeq 1-\sqrt{1-z[1+Q(v)-Q_0(v)]}. \qquad (47)$$

Expanding the expression on the right-hand side in powers of z we arrive at

$$Q_m(v)\simeq P_m[1+Q(v)-Q_0(v)]^m, \qquad (48)$$

with P_m the size distribution (32). The cluster size-velocity distribution is obtained by differentiating the auxiliary distribution $Q_m(v)$

$$P_m(v)\simeq mP_m[P_0(v)-P(v)][1+Q(v)-Q_0(v)]^{m-1} \quad (49)$$

Similar to the velocity distribution and the relaxation scales, the size velocity distribution as well can be obtained explicitly from the auxiliary function $Q(v)$. Consequently, it is characterized by a boundary layer structure. The size-velocity distribution is characterized by an exponential dependence upon the size, with a velocity dependent prefactor. Additionally, there is an algebraic prefactor that characterizes the overall size distribution.

The detailed analysis of the cluster size distribution suggests that these results apply only for sufficiently small sizes. Eqs. (47)-(49) should hold as long as the (dropped) term $R^{-1}[mQ_{m+1}(v,t)-(m-1)Q_m(v,t)]$ is negligible compared with the (kept) term $c_\infty Q_m$. Using (48), the above approximation is valid when

$$m\ll\sqrt{R}\,[Q_0(v)-Q(v)]\,. \qquad (50)$$

Hence, the range of validity of Eq. (49) strongly depends on the cluster velocity. This can be seen using the average cluster size $\langle m(v)\rangle = G(v)/P(v) = \sum_m mP_m(v)/\sum_m P_m(v)$, given by Eq. (25). Estimating the same quantity from Eq. (49), gives the correct leading large R behavior when $v\gg v^*$, while it gives a diverging average size rather than $\langle m(v)\rangle\sim R$ when $v\to 0$. Indeed, the condition (50) is satisfied by the $\langle m(v)\rangle$ only outside the boundary layer. Therefore the approximate cluster size-velocity distribution is useful for small and average sizes when $v\gg v^*$, while it holds only for sufficiently small sizes when $v\ll v^*$. Obtaining the large size tail requires a more detailed analysis similar to that performed for the size distribution.

VII. SUMMARY AND OUTLOOK

In summary, we introduced an approximation method for analyzing the Boltzmann equations for one dimensional traffic flows. In analogy with the Maxwell model (MM) of kinetic theory, we assumed a constant collision rate. This approach results in first order (in the velocity) differential equations. Analysis of these equations leads to explicit expressions for time dependent velocity distributions. Size-velocity distributions can be determined in the steady state as well. Although there are some quantitative deviations, the overall qualitative behavior including a boundary layer structure, existence of laminar and congested phases, etc. is in agreement with the results of the original Boltzmann equations. Several quantities such as the size growth exponent 1/2 actually agrees with the Boltzmann equation. We conclude that overall, the Maxwell approach is faithful to nature of the problem and thus provides a useful approximation scheme.

The MM allows explicit calculation of several important features, which are otherwise difficult to obtain. The approach towards the steady state is generally exponential and is characterized by a wide spectrum of velocity dependent relaxation scales, the smallest of which corresponds to the overall cluster density. The steady state size distribution exhibits an unusual scaling form with a scaling variable $m/\langle m\rangle^2$ rather than $m/\langle m\rangle$ which is naively expected. Additionally, the typical size which is of order unity is much smaller than the average size which grows with the collision number. This is a consequence of the algebraically diverging distribution of small sizes. This is an outcome of the non-equilibrium nature of the steady state that does not satisfy detailed balance as passing events reduce the cluster size by one only, while clustering events can increase the cluster size by a large number. This feature is independent of the details of the collision mechanism and we expect the most features underlying the size distribution to hold generally.

The MM can be refined and systematically improved. Some of the quantitative disagreements between the Maxwell and Boltzmann equation are rather obvious. For example the correct value of the crossover velocity can be obtained by replacing the integral $\int_0^v dv' P_0(v')$ with the integral $\int_0^v dv'(v-v')P_0(v')$. This compensates for the approximate kernel taken in the MM and results in the correct scaling exponents for the crossover velocity in both passing and no passing zones.

Furthermore, an appropriate choice of the value of the prefactor u_0 reduces the discrepancies between the two approaches. For example, the MM gives a universal dependence of the density upon the collision number, $c\sim R^{-1/2}$. However, for the BE if one assumes an algebraic intrinsic distribution near the origin, $P_0(v)\sim v^\mu$ as $v\to 0$ different behaviors are found for positive and negative μ [25]. For $\mu>0$, the density exhibits the universal behavior, $c_\infty\sim R_0^{-1/2}$, while for $\mu<0$ the density becomes μ dependent, $c_\infty\sim R_0^{-(\mu+1)/(\mu+2)}$. Here $R_0=c_0v_0t_0$ is the collision number within the Boltzmann framework. Choosing $u_0=\langle v\rangle = R_0^{\mu/(\mu+2)}$ (the actual BE behavior) implies $R = \nu R_0 \sim R_0^{(2\mu+2)/(\mu+2)}$, and thence $c_\infty \sim R_0^{-(\mu+1)/(\mu+2)} \sim R^{-1/2}$. Therefore the BE and MM results are consistent with each other if the appropriate choice for the collision rate $u_0 = \langle v\rangle$ is made.

Additionally, it would be interesting to compare the MM with the actual traffic process. Although the BE description is plausible at the steady state, it is clearly an approximation for the transient regime. For example, the BE differs from the exact behavior in the no passing case. Another avenue for further research is inhomogeneous traffic flows where a hydrodynamic description may prove useful. The hydrodynamic framework should involve a multicomponent fluid parametrized by the cluster size m. Specifically, the macroscopic description requires the density $P_m(x,t)$, the average velocity $u_m(x,t) = P_m^{-1}(x,t)\int dv\, v P_m(v,x,t)$, and the "temperature" (the average velocity square) for each m. Such infinite-fluid hydrodynamics may be quite different from the conventional one-fluid hydrodynamics. Indeed, Eq. (25) shows that the velocity decreases as the mass increases. Similar results apply for the temperature and thus, the equipartition of "energy" breaks down as well, in contrast with usual hydrodynamics.

In conclusion, the MM is a useful approximation to the kinetic traffic equations. This approach may be applicable to other traffic problems as well. In particular, it will be interesting to apply the Maxwell approach to traffic models with more realistic passing mechanisms.

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APPENDIX A: THE CONDITIONAL VELOCITY DISTRIBUTION

The car velocity distribution involves the leading car as well as the rest of slowed down cars in the cluster. The former is described by the cluster velocity distribution, while the latter is represented by $P(v, v', t)$, the density of cars of intrinsic velocity v driving with velocity v'. For the Maxwell model, the master equation for this conditional distribution reads

$$\frac{\partial P(v,v',t)}{\partial t} = -R^{-1}P(v,v',t) + P(v,t)P(v',t) + P(v',t)\int_{v'}^{v} dv'' P(v,v'',t) - P(v,v',t)\int_0^{v'} dv'' P(v'',t). \tag{A1}$$

The first term accounts for loss due to escape, while the rest of the terms represent changes due to collisions. Integrating these equations over the first velocity index and using the relation $G(v) = P(v) + \int_v^\infty dw\, P(w,v)$, one indeed recovers the rate equation (18).

Let us introduce the auxiliary function $Q(v,v',t) = \int_{v'}^{v} dw\, P(v,w)$ which gives the conditional velocity distribution by differentiation $P(v,v',t) = -\partial Q(v,v',t)/\partial v'$. This auxiliary function evolves according to

$$-\frac{\partial}{\partial t}\frac{\partial Q(v,v',t)}{\partial v'} = Q(v',t)\,\frac{\partial Q(v,v',t)}{\partial v'} + P(v,t)\,\frac{\partial Q(v',t)}{\partial v'} + Q(v,v',t)\,\frac{\partial Q(v',t)}{\partial v'}. \tag{A2}$$

Integrating Eq. (A2) over v' and using the boundary condition $Q(v,v,t) = 0$, we get

$$-\frac{\partial Q(v,v',t)}{\partial t} = Q(v',t)Q(v,v',t) + P(v,t)Q(v',t) - P(v,t)Q(v,t). \tag{A3}$$

This is a linear inhomogeneous differential equation for the auxiliary function $Q(v,v',t)$ which includes already known cluster velocity distributions. Integrating Eq. (A3) we arrive at

$$Q(v,v',t) = \int_0^t dt'\, P(v,t')[Q(v,t') - Q(v',t')]\ \exp\left[-\int_{t'}^{t} dt''\, Q(v',t'')\right]. \tag{A4}$$

The exact solution (A4) can in principle be reduced to a more explicit expression by following the procedure detailed in Appendix B for transforming the formal solution of Eq. (B1) into Eq. (B2). Such a solution is very cumbersome so we do not give it here.

The steady state conditional distribution is obtained immediately from Eq. (A3) $Q(v,v') = P(v)[Q(v)/Q(v') - 1]$. The joint distribution is found by differentiation, $P(v,v') = -\partial Q(v,v')/\partial v' = P(v)P(v')Q(v)/Q^2(v')$, or explicitly

$$P(v,v') = \frac{RP_0(v)P_0(v')}{[1+2RI_0(v')]^{3/2}}. \tag{A5}$$

In the laminar phase, this conditional distribution is proportional to R, while it is algebraically suppressed in the congested phase. One can verify that Eq. (A5) is consistent with the cluster and car distributions using the relations $P(v) = P_0(v) - \int_0^v dv' P(v,v')$, and $G(v) = P(v) + \int_v^\infty dw\, P(w,v)$, respectively.

APPENDIX B: THE AUXILIARY CAR VELOCITY DISTRIBUTION

The master equation (20) for $g(v,t)$ can be integrated formally

$$g(v,t) = g_0(v)\left[\exp\left(-\int_0^t dt'\, Q(v,t')\right) + R^{-1}\int_0^t dt'\, \exp\left(-\int_{t'}^{t} dt''\, Q(v,t'')\right)\right]. \tag{B1}$$

To obtain more explicit results, we notice that the velocity plays the role of a parameter in Eq. (B1). We thus change variable from t' to $q \equiv Q(v,t')$ and for example

$$\int_0^t dt'\, Q(v,t') = -2\int_{Q_0}^{Q} dq\, \frac{q}{q^2 - Q_\infty^2} = \ln\frac{Q_0^2 - Q_\infty^2}{Q^2 - Q_\infty^2}.$$

This change of variables allows us to perform the integration

$$
\begin{aligned}
g(v,t) &= g_0 \left[\frac{Q^2 - Q_\infty^2}{Q_0^2 - Q_\infty^2} - 2\frac{Q^2 - Q_\infty^2}{R} \int_{Q_0}^{Q} \frac{dq}{(q^2 - Q_\infty^2)^2} \right] \\
&= g_0 \left[\frac{Q^2 - Q_\infty^2}{Q_0^2 - Q_\infty^2} + \frac{Q}{RQ_\infty^2} - \frac{Q_0}{RQ_\infty^2} \frac{Q^2 - Q_\infty^2}{Q_0^2 - Q_\infty^2} + \frac{Q^2 - Q_\infty^2}{2RQ_\infty^3} \ln \left(\frac{Q - Q_\infty}{Q + Q_\infty} \Big/ \frac{Q_0 - Q_\infty}{Q_0 + Q_\infty} \right) \right] \\
&= g_0 \left[\frac{Q}{RQ_\infty^2} + \frac{Q^2 - Q_\infty^2}{I_0^2} \left(1 - \frac{Q_0}{RQ_\infty^2} \right) + \frac{Q^2 - Q_\infty^2}{2RQ_\infty^3} \ln(e^{-tQ_\infty}) \right] \\
&= \frac{g_\infty}{Q_\infty} \left[Q + (Q^2 - Q_\infty^2) \left(\frac{1}{I_0} - \frac{t}{2} \right) \right] . \qquad \text{(B2)}
\end{aligned}
$$

In the above derivation we used the identities $g_\infty = g_0/RQ_\infty$, $Q_0^2 - Q_\infty^2 = I_0^2$, and $[1 - Q_0/RQ_\infty^2] = I_0/RQ_\infty^2$.