Popularity-Driven Networking

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We investigate the growth of connectivity in a network. In our model, starting with a set of disjoint nodes, links are added sequentially. Each link connects two nodes, and the connection rate governing this random process is proportional to the degrees of the two nodes. Interestingly, this network exhibits two abrupt transitions, both occurring at finite times. The first is a percolation transition in which a giant component, containing a finite fraction of all nodes, is born. The second is a condensation transition in which the entire system condenses into a single, fully connected, component. We derive the size distribution of connected components as well as the degree distribution, which is purely exponential throughout the evolution. Furthermore, we present a criterion for the emergence of sudden condensation for general homogeneous connection rates.

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Networks are sets of nodes that are connected by links. Models for the evolution of complex networks fall into two general classes: *evolving graphs* where the number of nodes is fixed but the number of links grows, and *growing networks* where the number of nodes and the number of links both grow [1–7]. The classic evolving random graph model, where pairs of randomly selected nodes are repeatedly connected by links, captures the nucleation of a giant connected component with a macroscopic number of nodes [8, 9]. The preferential attachment model of network growth, where newly added nodes connect to existing nodes with probability that is proportional to the degree, yields the broad degree distributions with power-law tails that characterize many real-world complex networks [10, 11].

In this letter, we study random graphs that evolve according to a preferential attachment mechanism. In our model, the number of nodes is fixed but the number of connections grows. The "popularity" of each node, as measured by the degree, governs the connection process. This fusion between the two seminal network models of random graphs and preferential attachment, is inspired by Facebook, the immense online network of cyber-friends [6]. In Facebook, new connections are formed via friendship requests from one member of the network to another; upon acceptance, the two become friends. Preferential attachment implies that members seek and accept friends according to popularity.

In our model for the popularity-driven growth of connectivity in a network, the system consists initially of a set of disjoint nodes. We consider the natural situation where the degree of a node, defined as the number of existing connections, controls the rate by which two nodes connect with each other. Specifically, a node with degree i and a node with degree j connect at a rate that is linear in i and linear in j as well. We find that the degree distribution remains purely exponential throughout the evolution. Thus, the preferential attachment mechanism leads to opposite results in fixed and in growing networks: the tail of the degree distribution is narrow in the former but broad in the latter.

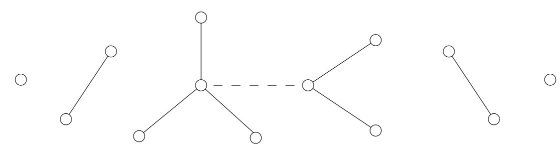

FIG. 1: Connection and subsequent aggregation. The new link (dashed line) connects two nodes with degrees $i = 3$ and $j = 2$. As a consequence, two connected components with sizes $l = 4$ and $m = 3$ merge.

Our main result is that the system undergoes two finite-time transitions. In the first transition, occurring at time t_g, a giant connected component nucleates. The giant component contains a finite fraction of all nodes. In the second condensation transition, the giant component takes over the entire system. In contrast with classical random graphs, the entire network condenses into a single component at a finite time t_c. We obtain analytically the degree distribution and the size distribution of connected components when the rate of connection between nodes with degree i and nodes with degree j equals the product $(i+1)(j+1)$. From these two distributions, we obtain the finite transition times:

$$t_g = 1/3 \qquad \text{and} \qquad t_c = 1. \tag{1}$$

The rest of this letter includes two parts. In the first, we study how the degree of a node grows as a result of the networking process, and in the second, we investigate how the size of a connected component increases as a result of the very same process.

In our evolving random graph model, the network consists of a set of disjoint nodes at time $t = 0$. The network becomes connected through the sequential addition of links: a node with degree i and a node with degree j connect with rate $C_{i,j}$. As a result, the degree undergoes the augmentation process (figure 1)

$$(i, j) \xrightarrow{C_{i,j}} (i+1, j+1). \tag{2}$$

Since the two nodes are interchangeable, the connection rate is symmetric, $C_{i,j} = C_{j,i}$. We note that the classic random graph corresponds to the uniform connection rate $C_{i,j} = 1$ [8, 9].

Throughout this study, we implicitly take the infinite system size limit. Let $n_j(t)$ be the degree distribution, that is, the fraction of nodes with degree j at time t. This quantity is normalized, $\sum_j n_j = 1$, and it obeys the rate equation

$$\frac{dn_j}{dt} = \nu_{j-1} n_{j-1} - \nu_j n_j. \tag{3}$$

The initial condition is $n_j(0) = \delta_{j,0}$. The quantity ν_j equals the total connection rate of nodes with degree j, $\nu_j = \sum_i C_{i,j}\, n_i$.

By summing the evolution equations (3), we can verify that the normalization is preserved, $(d/dt)\sum_j n_j = 0$. Similarly, by multiplying (3) with j and summing over all j, we find that the average degree, $\langle j\rangle = \sum_j j n_j$, grows according to

$$\frac{d\langle j\rangle}{dt} = \sum_{i,j} C_{i,j} n_i n_j. \tag{4}$$

This equation reflects the connection process (2). Since every link connects two nodes, the average degree conveniently yields the total density of links, L, via the simple relation $2L = \langle j\rangle$.

In the preferential attachment model of network growth, the attachment rate is linear in the degrees of the existing nodes. In this "rich-get-richer" mechanism, the connection probability is proportional to the popularity. Hence, to model popularity-driven networking in evolving graphs, we restrict our attention to rates that are linear in both i and j, $C_{i,j} = (i+a)(j+a)$. Often used in preferential attachment, the offset a allows us to start with a set of disjoint nodes, the natural initial condition. Here, we investigate the rate

$$C_{i,j} = (i+1)(j+1). \tag{5}$$

We verified that the qualitative behavior, including the the sudden condensation transition, extends to all a [12].

By substituting the rate (5) into equation (4), we see that the average degree obeys the closed equation

$$\frac{d\langle j\rangle}{dt} = \big(\langle j\rangle + 1\big)^2. \tag{6}$$

We obtain the average degree by solving this equation, subject to the initial condition $\langle j\rangle\big|_{t=0} = 0$,

$$\langle j\rangle = \frac{t}{1-t}, \tag{7}$$

when $t < 1$. Hence, the average degree diverges in finite time.

From the average degree (7), we find that the connection rate ν_j is linear in the degree, $\nu_j = (j+1)(1-t)^{-1}$. Therefore, the degree distribution obeys the linear evolution equation

$$(1-t)\frac{dn_j}{dt} = j\, n_{j-1} - (j+1)\, n_j. \tag{8}$$

We solve these equations recursively, starting with the initial condition $n_j(0) = \delta_{j,0}$, and obtain $n_0 = 1-t$, $n_1 = (1-t)t$, $n_2 = (1-t)t^2$, etc. Generally one finds that the degree distribution is purely exponential [5],

$$n_j = (1-t)\, t^j. \tag{9}$$

Surprisingly, the degree distribution vanishes in finite time: $n_j(t_c) = 0$ for all j with $t_c = 1$. Consequently, the degree of a node is finite if and only if $t < t_c$. This behavior is a signature of the continuous condensation transition that occurs at the condensation time t_c. As shown below, the entire network condenses into a single, fully connected, component at this time.

The difference between growing and fixed networks is remarkable. Networks that grow by preferential attachment have broad degree distributions with power-law tails [10, 11, 13–15], but in a fixed network, preferential attachment leads to a narrow degree distribution with exponential tail. On its own, the rich-get-richer mechanism does not generate broad tails, but rather, it is the combination of a growing network and preferential attachment that leads to a broad distribution of degrees. We note that exponential degree distributions occur in a variety of complex networks including power, transportation, and social communication networks [16, 17].

Our evolving graph consists of multiple connected components ("clusters") which undergo binary aggregation as a result of the connection process (2). Symbolically, such an aggregation process [18, 19] can be represented as (figure 1)

$$[l] + [m] \xrightarrow{K_{l,m}} [l+m], \tag{10}$$

where l and m are the number of nodes in the two merging clusters. The rate of aggregation between a *finite* cluster of size l and a *finite* cluster of size m is

$$K_{l,m} = (3l-2)(3m-2). \tag{11}$$

To obtain this rate, we note that a cluster with k nodes has $k-1$ links. (This relation is valid for clusters with tree structure, and indeed, nearly all finite clusters are trees in the infinite size limit [7].) Equation (11) follows from the connection rate (5), together with the fact that the sum of the degrees in the cluster equals twice the number of links.

The total density of clusters, c, follows from the total density of links, L, when all clusters are finite. Since every link reduces the number of clusters by one, we have $c(t) = 1 - L(t)$. Using $2L = \langle j\rangle$, we conclude $c = 1 - \langle j\rangle/2$, and from (7), we obtain the cluster density

$$c(t) = \frac{2-3t}{2(1-t)}\,. \tag{12}$$

As shown below, this relation holds when $t < t_g$ with t_g given in (1).

The density $c_k(t)$ of clusters with size k at time t obeys the master equation

$$\frac{dc_k}{dt} = \frac{1}{2}\sum_{l+m=k}(3l-2)(3m-2)c_l c_m - \langle j+1\rangle(3k-2)c_k, \quad (13)$$

where $\langle j+1\rangle = (1-t)^{-1}$. The initial condition is $c_k(0) = \delta_{k,1}$. The gain term is *nonlinear*, and it directly reflects the aggregation process (10) with the product rate (11). The loss term, on the other hand, is *linear* in the cluster-size density. The loss rate equals the product between the sum of all degrees (shifted by one) in the *cluster*, and the average degree (shifted by one) of all nodes in the entire *system*. In this form, the master equation is valid at all times, whether all clusters are finite or whether macroscopic clusters exist as well [20].

Let $M_n = \sum_n k^n c_k$ be the nth order moment of the size distribution. The zeroth moment gives the total cluster density, $M_0 \equiv c$, and the first moment, M_1, yields the total mass of finite components. Furthermore, the second moment satisfies

$$\begin{aligned}\frac{dM_2}{dt} &= (3M_2 - 2M_1)^2 \\ &+ (3M_3 - 2M_2)\left[(3M_1 - 2M_0) - \langle j+1\rangle\right],\end{aligned}$$

subject to the initial condition $M_2(0) = 1$. We obtain this equation by multiplying (13) by k^2 and summing over all k. Let's assume that finite clusters contain all the mass, $M_1 = 1$. In this case, the cluster density (12) gives $3M_1 - 2M_0 = (1-t)^{-1}$, and as a consequence, the second moment satisfies the closed equation $dM_2/dt = (3M_2 - 2)^2$. Therefore,

$$M_2 = \frac{1-2t}{1-3t}, \quad (14)$$

for $t < 1/3$. The divergence of the second moment shows that a percolation transition [21] occurs at time $t_g = 1/3$ as stated in (1). The critical density of links $L_g = L(t_g) = 1/4$ is smaller than the value $L_g = 1/2$ corresponding to the classic random graph [9].

When $t < t_g$, the system is in a non-percolating phase and finite clusters contain all the mass, $M_1 = 1$. Otherwise, the system is in a percolating phase, where a macroscopic cluster, the giant component [22], contains a finite fraction of all nodes and $M_1 < 1$.

To find the mass of the giant component, we analyze the cluster-size distribution. First, we solve (13) recursively for small clusters. We can confirm that the density of minimal-size clusters ("monomers") equals the fraction of isolated nodes, $c_1 = n_0 = 1 - t$. The densities of "dimers" and "trimers" are $c_2 = \frac{1}{2}t(1-t)^3$ and $c_3 = t^2(1-t)^5$. We therefore expect the general form

$$c_k = A_k\, t^{k-1}(1-t)^{2k-1}. \quad (15)$$

By substituting this expression into (13), we obtain a recursion equation for the coefficients A_k

$$2(k-1)A_k = \sum_{l+m=k}(3l-2)(3m-2)A_l A_m, \quad (16)$$

for $k > 1$, subject to $A_1 = 1$. We now make the transformation $B_k = (3k-2)A_k$. The coefficients B_k satisfy a second recursion relation

$$2(k-1)B_k = (3k-2)\sum_{l+m=k} B_l B_m,$$

for $k > 1$, subject to $B_1 = 1$. From this recursion, we conclude that the generating function $B(x) = \sum_k B_k\, x^k$, obeys the differential equation $x\frac{dB}{dx} = \frac{B(1-B)}{1-3B}$. We now integrate this differential equation subject to the constraint $\lim_{x\to 0} x^{-1}B = 1$, and find that $B(x)$ satisfies the cubic equation

$$B(1-B)^2 = x. \quad (17)$$

We obtain the coefficients B_k by using the Lagrange inversion method,

$$\begin{aligned}B_k &= \frac{1}{2\pi i}\oint \frac{B}{x^{k+1}}dx \\ &= \frac{1}{2\pi i}\oint \frac{B(1-B)(1-3B)}{B^{k+1}(1-B)^{2(k+1)}}dB \\ &= \frac{1}{2\pi i}\oint \frac{1-3B}{B^k(1-B)^{2k+1}}dB \\ &= \frac{1}{2\pi i}\oint \left[\sum_{n=0}^{\infty}\binom{2k+n}{n}\left(B^{n-k} - 3B^{n+1-k}\right)\right]dB \\ &= \binom{3k-1}{k-1} - 3\binom{3k-2}{k-2} = \frac{(3k-2)!}{k!(2k-1)!}.\end{aligned}$$

In the second line, we used equation (17), and replaced the integration over x with integration over B, by using $dx = (1-B)(1-3B)dB$. Finally, we utilize the identity $(1-B)^{-m} = \sum_{n=0}^{\infty}\binom{m+n-1}{n}B^n$, to determine the coefficients A_k,

$$A_k = \frac{(3k-3)!}{k!(2k-1)!}. \quad (18)$$

The integer sequence $2A_k = \{2, 1, 2, 6, 22, 91, \ldots\}$ also arises in planar maps [23, 24].

The tail of the cluster-size distribution is a product of a power law and an exponential,

$$c_k \simeq \frac{1}{\sqrt{12\pi}}\, k^{-5/2}\, e^{-k/k_*}, \quad (19)$$

for $k \to \infty$, with $k_*^{-1} = \ln\frac{t_g(1-t_g)^2}{t(1-t)^2}$. This tail follows from the Stirling formula and the coefficients (18). Hence, at the percolation transition, the distribution is purely algebraic, $c_k(t_g) \sim k^{-5/2}$ [25]. Otherwise, the tail is power law at small scales, $k \ll k_*$, but exponential at

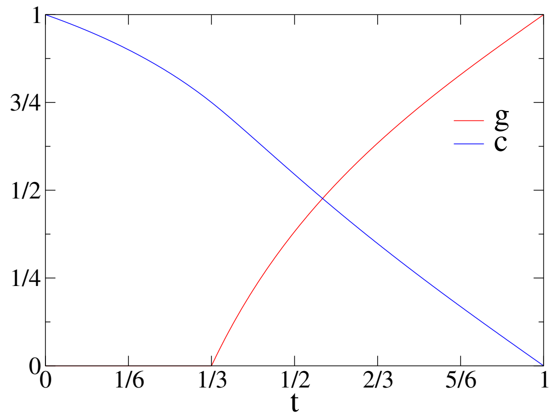

FIG. 2: The mass of the giant component, g, and the total density of connected components, c, versus time t.

large scales, $k \gg k_*$. The characteristic scale diverges, $k_* \simeq \frac{4}{27}(t-t_g)^{-2}$, as $t \to t_g$.

Note that the time dependence of the cluster-size density (15) enters primarily through the quantity $t(1-t)^2$ as $c_k \propto [t(1-t)^2]^k$. This observation tells us that for an auxiliary time τ, that is connected to the physical time t via the cubic equation

$$\tau(1-\tau)^2 = t(1-t)^2, \tag{20}$$

the densities $c_k(t)$ and $c_k(\tau)$ are related by the duality relation

$$c_k(t) = c_k(\tau)\left(\frac{1-t}{1-\tau}\right). \tag{21}$$

When $t < t_g$, equation (20) has only the trivial solution $\tau = t$, while for $t_g < t < t_c$ there is additionally a second nontrivial root $\tau < t_g$. Consequently, for all $t_g < t < t_c$ we can choose the nontrivial root $\tau < t_g$ of (20) and then, the duality relation (21) conveniently specifies the cluster-size distribution in the percolating phase in terms of its counterpart in the non-percolating phase. By multiplying (21) by k and summing over all k we see that $M_1(t)/(1-t) = M_1(\tau)/(1-\tau)$, and since $M_1(\tau) = 1$, we immediately get the nontrivial first moment in the percolating phase, $M_1 = (1-t)/(1-\tau)$. In summary, the mass of finite clusters is

$$M_1(t) = \begin{cases} 1 & 0 \le t \le t_g, \\ \frac{1-t}{1-\tau} & t_g \le t \le t_c, \\ 0 & t \ge t_c. \end{cases} \tag{22}$$

The mass of the giant component, g, equals the complementary mass, $g = 1 - M_1$ (see Fig. 2). Post-percolation, this quantity grows linearly, $g(t) \simeq 3(t-t_g)$, as $t \downarrow t_g$, and furthermore, $1 - g(t) \simeq t_c - t$ as $t \uparrow t_c$. At time $t_c = 1$, the giant component takes over the entire graph, and hence, the system condenses into a single component. Remarkably, the cluster-size density and the fraction of nodes with finite degrees vanish *simultaneously*, $c_k(t_c) = n_j(t_c) = 0$.

For completeness, we mention that the generating function, $\mathcal{C}(z,t) = \sum_k c_k(t)\, z^k$, can be written as an explicit function of time using the hypergeometric function,

$$\mathcal{C}(z,t) = \frac{1}{3t(1-t)}\left[F\left(-\tfrac{2}{3}, -\tfrac{1}{3}; \tfrac{1}{2}; \tfrac{3^3}{2^2} t(1-t)^2\, z\right) - 1\right].$$

Moments of the distribution can also be written explicitly [26]. The density $c(t) \equiv \mathcal{C}(z=1,t)$ is plotted in figure 2, and one can verify that this quantity matches (12) in the non-percolating phase.

An unusual feature of our network connection process is that both the percolation transition and the condensation transition occur at finite times. Let us consider the uniform connection rate $C_{i,j} = 9$, corresponding to the classic random graph. For this evolving graph, the percolation time is finite, $t_g = 1/9$, but the condensation time is divergent, $t_c = \infty$ (in a finite system, the condensation time is logarithmic in the total number of nodes) [7]. Interestingly, the corresponding aggregation rate, $K_{l,m} = (3l)(3m)$, is *larger* than the aggregation rate (11), yet the latter, smaller, rate produces faster condensation! Of course, the aggregation rate $K_{l,m}$ corresponds only to *finite* clusters, and it does not apply to the giant component. The rich-get-richer mechanism accelerates [27, 28] the rate by which the giant component engulfs finite components, and this feature is ultimately responsible for the finite-time condensation.

We can also evaluate the number of links required to achieve percolation in a finite system with N nodes. For such a system, condensation occurs at time $t_c(N)$ with $1-t_c \sim N^{-1}$. To obtain this estimate, we simply consider the time when the last isolated node joins the giant component, $Nn_0(t_c) \sim 1$ and use $n_0 = 1-t$. The total number of links $L_{\rm tot} = N\langle j\rangle_c/2 \sim N(1-t_c)^{-1}$ therefore grows quadratically with the size of the system, $L_{\rm tot} \sim N^2$. Hence, the total number of links needed for condensation in a popularity-driven network is much larger than in ordinary random graphs for which $L_{\rm tot} \sim N \ln N$ [7].

We also considered a general class of evolving graphs with homogeneous connection rates, and we now briefly discuss the purely algebraic rates

$$C_{ij} = (ij)^\alpha. \tag{23}$$

The case $\alpha = 0$ corresponds to ordinary random graphs, and the case $\alpha = 1$, to popularity-driven networking. From (4), the average degree grows according to

$$\frac{d\langle j\rangle}{dt} \sim \langle j^\alpha\rangle^2 \sim \langle j\rangle^{2\alpha}, \tag{24}$$

where we assumed the scaling behavior $\langle j^\alpha\rangle \sim \langle j\rangle^\alpha$. From this rate equation, we find the scaling behaviors

$$\langle j\rangle \sim \begin{cases} t^{1/(1-2\alpha)} & \alpha < 1/2, \\ e^{\text{const.}\times t} & \alpha = 1/2, \\ (t_c - t)^{-1/(2\alpha-1)} & 1/2 < \alpha \le 1. \end{cases} \tag{25}$$

When $\alpha < 1/2$, the average degree grows algebraically with time. Furthermore, the percolation time is finite, but the condensation time is infinite. In the marginal case $\alpha = 1/2$, the degree grows exponentially with time. When $1/2 < \alpha \le 1$, the percolation time is finite, and the condensation time is finite as well. The average degree diverges as the condensation transition is approached.

Finally, when $\alpha > 1$, the scaling assumption used in (24) is no longer valid, and condensation becomes instantaneous: $t_c = t_g = 0$ [29]. This phenomenon is analogous to the instantaneous gelation that occurs in aggregation [30, 31] and exchange-driven growth [32].

In conclusion, we studied the growth of connectivity in an evolving graph. In our model, the degree controls the connection process as the probability that a node forms a new connection is proportional to the total number of its existing connections. We find that the system undergoes two continuous transitions, both occurring at finite times. In the first percolation transition, a macroscopic connected component nucleates. In the second condensation transition, the network becomes fully connected. We obtained analytically the degree distribution and the size distribution of connected components. We find that in a fixed network, the degree distribution is exponential, and therefore, the average degree fully characterizes the entire distribution.

Our theoretical analysis relies heavily on the fact that the connection rate is linear in the degree. For such connection rates, the rate by which components merge is a bilinear function of size. Consequently, the size distribution of connected components is analytically tractable. Linear connection rates also have computational advantages. Popularity-driven networking can be studied using a convenient *random link algorithm*. In this numerical simulation, we choose two links at random. For each link we pick one of the two nodes connected to it, and then, connect these two nodes. With this implementation, we select nodes with probability that is proportional to the degree [7], and this procedure can be generalized to arbitrary linear rates. The random link algorithm has optimal efficiency as the computational cost is linear in the number of links, and it can be used to study additional properties of our evolving graph. Among a host of possibilities, we mention structural properties of connected components such as their cycle structure [22], and extremal properties such as statistics of the largest connected component.

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