Opinion dynamics: rise and fall of political parties

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We analyze the evolution of political organizations using a model in which agents change their opinions via two competing mechanisms. Two agents may interact and reach consensus, and additionally, individual agents may spontaneously change their opinions by a random, diffusive process. We find three distinct possibilities. For strong diffusion, the distribution of opinions is uniform and no political organizations (parties) are formed. For weak diffusion, parties do form and furthermore, the political landscape continually evolves as small parties merge into larger ones. Without diffusion, a pattern develops: parties have the same size and they possess equal niches. These phenomena are analyzed using pattern formation and scaling techniques.

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Interacting particle systems and agent-based models are becoming increasingly important in the behavioral, social, and political sciences [1–3]. There is compelling evidence that collective phenomena emerging in social contexts can be attributed to basic agent-agent interactions [4]. Statistical physics and nonlinear dynamics methods naturally apply for analysis of simplified interacting particle systems [5]. Paradigms such as scaling, criticality, and universality, emerging from such quantitative analysis, can guide and validate detailed agent-based models, used to simulate real-world situations.

In opinion dynamics, recent studies focus on the emergence of cooperative phenomena including spatial organization, the formation of coherent structures (political parties), and the transition from unity to discord [5–10]. In particular, the remarkably simple compromise process, in which pairs of agents reach a fair compromise, captures familiar political systems: one-party, two-party, etc, [11–17].

This investigation generalizes the compromise process by allowing individual agents to change their opinions in a random, diffusive fashion. It is shown that diffusion is an essential element of opinion dynamics. It generates realistic lifecycles of political organizations and additionally, it governs the transition from a disorganized to an organized political system.

The competition between compromise and diffusion is quantified by one parameter, the diffusion constant. For strong randomness, the political system is disorganized and the distribution of opinions is uniform. For weak randomness, political organizations do form and they evolve constantly. Large parties overtake smaller ones and the separation between neighboring parties grows indefinitely. Without randomness, a stationary pattern with evenly-spaced, evenly-sized parties forms.

In our model, the opinion of an agent is quantified by an integer n, and it changes via two separate processes. The first is compromise. Two randomly selected agents reach consensus, provided that their opinion difference is smaller than a fixed threshold, set to two for simplicity,

$$(n-1, n+1) \xrightarrow{1} (n, n). \tag{1}$$

The compromise process occurs at a constant rate, set to 1. The second process is diffusion. An agent may change his or her opinion in a random fashion,

$$n \xrightarrow{D} n \pm 1. \tag{2}$$

This is merely diffusion, a random walk in opinion space with D the diffusion constant. Of course, the total population is conserved. The total opinion is strictly conserved in compromise events, but it is conserved only on average for diffusive moves.

The compromise process mimics the human tendency for resolving conflicts [4], and the threshold incorporates a certain degree of conviction in one's own opinion. Diffusion accounts for the possibility that people may change their opinion either on their own or due to news events, editorials, etc.

The density $P_n(t)$ of agents with opinion n at time t obeys the master equation

$$\begin{aligned}\frac{dP_n}{dt} &= 2P_{n-1}P_{n+1} - P_n(P_{n-2} + P_{n+2}) \\ &+ D(P_{n-1} + P_{n+1} - 2P_n).\end{aligned} \tag{3}$$

The total population and the total opinion are conserved: $\sum_n P_n = \text{const}$ and $\sum_n nP_n = \text{const}$. Interactions between nearest neighbors were ignored because conservation of the total opinion implies that they do not contribute to the rate equations. We analyze in order one-party, two-party, and multi-party dynamics.

One-party dynamics. Consider the initial condition where m people have the opinion -1 and as many have opinion 0, $P_n(0) = m(\delta_{n,-1} + \delta_{n,0})$. This state corresponds to a well-defined political organization, namely, a party. Its size, equal to the total population, is taken to be large, $m \gg D$. Clearly, throughout the evolution, the opinion distribution remains symmetric, $P_n = P_{-1-n}$.

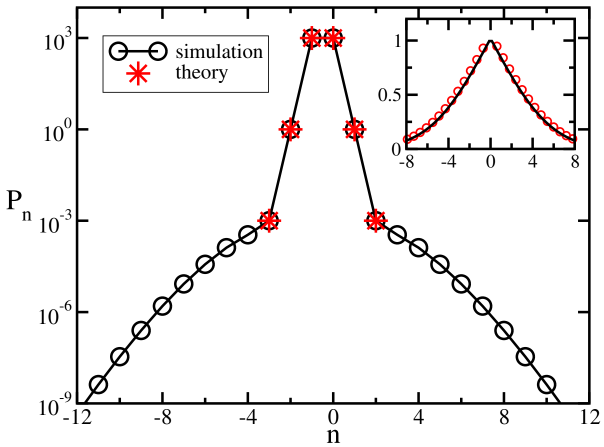

FIG. 1: One-party dynamics. Shown is $P_n(t=1)$ for $m = 10^3$. The theoretical prediction for the core (5) is shown for reference. The inset, where $\Phi(z)$ is plotted versus $z = nt^{-1/2}$, shows the scaling behavior of the tail at large times $t = 10^3$ (circles) and $t = 10^4$ (line).

Equation (3) supports the trivial steady-state where the opinion distribution vanishes, $P_n = 0$, as well as one in which compromise and diffusion balance

$$P_{n-1}P_{n+1} = DP_n. \tag{4}$$

Solving this equation recursively with $P_{-1} = P_0$ gives the periodic state $(P_{-1}, P_0, P_1, P_2, P_3, P_4) = (P_0, P_0, D, D^2/P_0, D^2/P_0, D)$, with $P_n = P_{n+6}$. Starting with the one-party initial condition, the distribution has a localized core that matches this periodic structure over a few lattice sites,

$$(P_0, P_1, P_2) \cong (m, D, D^2m^{-1}). \tag{5}$$

This is confirmed by numerical integration of (3), as shown in figure 1 [18]. Using the conservation law $P_0 + P_1 + P_2 \cong m$, Eq. (5) can be refined $(P_0, P_1, P_2) \cong (m-D, D, D^2m^{-1})$. The core is established very quickly: from the short time behavior, $P_n(t) \cong m\,(Dt)^n$, we immediately deduce the stabilization time scale m^{-1}. The larger the party, the faster it is shaped.

Outside the core, diffusion dominates over compromise since $P^2 \ll DP$ when $P \ll D$. The tail obeys the diffusion equation $dP_n/dt = D(P_{n-1} + P_{n+1} - 2P_n)$ for $n \geq 2$ with the boundary condition $P_2(t) = D^2m^{-1}$; this is the standard problem of diffusion with a source [19]. Consequently, the tail is characterized by the diffusive length scale $\ell \sim t^{1/2}$ and its shape is asymptotically self-similar

$$P_n(t) \to m^{-1}\Phi\left(n\,t^{-1/2}\right) \tag{6}$$

with $\Phi(0) = \text{const}$. Henceforth, the explicit dependence on the diffusion constant is dropped. The tail population, $\mu = 2\sum_{n\geq 2} P_n$, grows with time according to $\mu \sim m^{-1}t^{1/2}$. Eventually, the entire population is transferred from the core to the tail and the party dissolves. The lifetime of the party τ is estimated from $\mu \sim m$ to be

$$\tau \sim m^4. \tag{7}$$

This time scale grows rapidly with the party size, indicating that diffusive loss is negligible over a substantial period and that large parties are long-lived.

In summary, a single party has a quasi-stationary state consisting of a fixed, tightly confined core and an extended diffusive tail. Over its lifetime, the core of the party is immobile and it is unaffected by the random changes in position of its affiliates. The core contains the bulk of the population, its hight equals the party size and its depth is inversely proportional to the size. Ultimately, an isolated party dissolves. Its remnant is a diffusive cloud centered at the original party position with total population equal to the initial party size.

Two-party dynamics. To find out how two neighboring parties interact, we start start with opinions -3 and -2 of m_1 people each, plus opinions $l+2$ and $l+3$ with m_2 people each, $P_n = m_1(\delta_{n,-2}+\delta_{n,-3})+m_2(\delta_{n,l+2}+\delta_{n,l+3})$, corresponding to two large parties, $m_1, m_2 \gg D$, that are separated by a large distance $l \gg 1$.

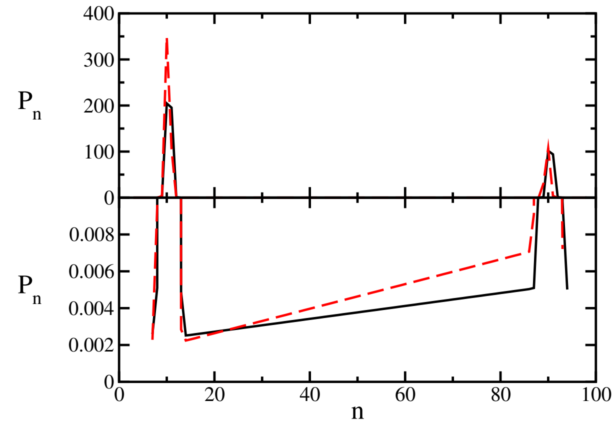

FIG. 2: Two-party dynamics. Shown are results of numerical integration with $m_1 = 2m_2 = 200$ and $l = 90$ at an early time $t = 10^3$ (solid lines) and a later time $t = 2 \times 10^5$ (dashed lines). The top plot shows the cores, and the bottom plot shows the tail.

Initially, the parties do not interact and each one follows the one-party dynamics above. When their diffusive tails meet, which occurs on the diffusive time scale l^2, the distribution reaches a steady-state in the region separating the two. It obeys the discrete Laplace equation $P_{n+1} + P_{n-1} - 2P_n = 0$ with the boundary conditions dictated by the two cores $P_0 \propto m_1^{-1}$ and $P_l \propto m_2^{-1}$. Therefore, there is a linear profile (figure 2)

$$P_n \propto \frac{1}{m_1} + \left(\frac{1}{m_2} - \frac{1}{m_1}\right)\frac{n}{l} \tag{8}$$

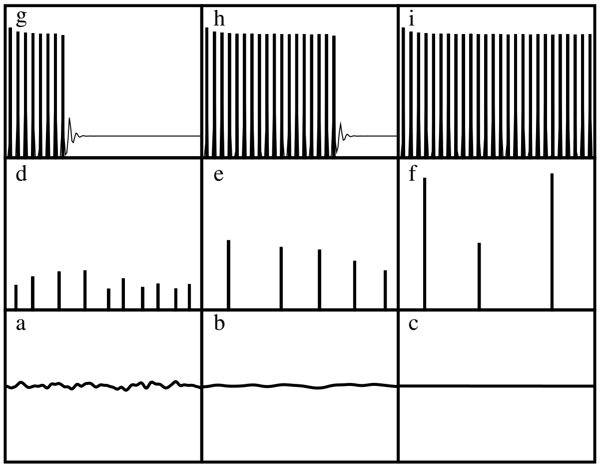

FIG. 3: Multi-party dynamics. Shown are representative snapshots for $D = 3$ (a-c), $D = 1$ (d-f), and $D = 0$ (g-i) at an early time (left), intermediate time (middle), and late time (right). Shown is P_n versus n using lines ($D = 3$, $D = 0$). Also shown is the total party size and position using bars with hight equal to the party size ($D = 1$, $D = 0$). The plots are results of numerical integration of (3) with $\epsilon = 0.1$ for $D \neq 0$ and $\epsilon = 0$ for $D = 0$.

for $0 \leq n \leq l$. As a result, there is a slow and steady flux from the smaller party into the larger one, $J = |P_{n+1} - P_n|$, or explicitly $J \propto l^{-1}(m_<^{-1} - m_>^{-1})$ with $m_< = \min(m_1, m_2)$ and $m_> = \max(m_1, m_2)$. We note that diffusion enables the two parties to interact. The flux is proportional to the difference in depth, and it is inversely proportional to the separation. Eventually, the small party is depleted. The depletion time can be estimated from the flux, $T \sim m_</J$, as

$$T \sim l\, m_<^2 \tag{9}$$

where the dependence on the larger population was tacitly ignored. An improved estimate for the depletion time can be obtained from the evolution equations $dm_>/dt = -dm_</dt = J$.

We conclude that there is a steady flux from the small party into its neighboring larger party resulting in the eventual demise of the smaller party. Thus, the result of the interaction is deterministic as it always leads to merger. The lifetime of the small party grows quadratically with its size. It also increases linearly with the separation or "niche". While the larger party grows during the merger, this growth does not affect its position; it is practically immobile.

Multi-party dynamics. The uniform state $P_n = \text{const}$ is stationary. Any uniform state can be transformed by an appropriate rescaling of the opinion distribution, time, and the diffusion constant into the state $P_n = 1$, so we address this case. To investigate the stability of the uniform state, we considered heterogeneous initial conditions with $P_n(0)$ a randomly chosen number in the range

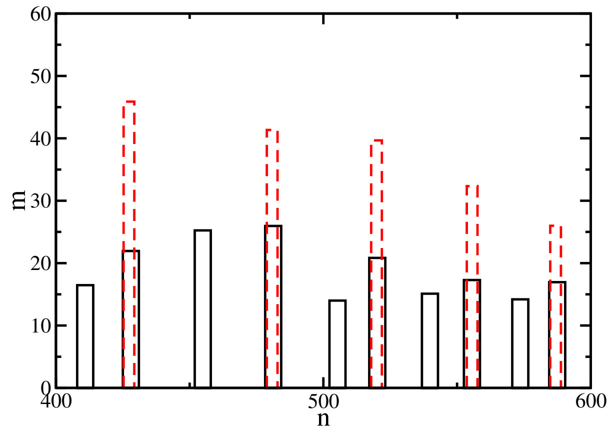

FIG. 4: The party size m versus position n. Shown are results of numerical integration of (3) with $D = 1$, $N = 10^3$, and $\epsilon = 0.1$ at times $t = 10^3$ (solid line) and $t = 10^4$ (dashed line). The bar hight equals the party size.

$[-1-\epsilon, 1+\epsilon]$ with $\epsilon \ll 1$. The system is large, $1 \leq n \leq N$ with $N \gg 1$.

Stability of the uniform state is studied using small periodic perturbations, $P_n = 1 + \phi_n$, with $\phi_n \propto e^{ikn+\lambda t}$. From (3), the growth rate is

$$\lambda = 2(2\cos k - \cos 2k - 1) + 2D(\cos k - 1). \tag{10}$$

The perturbation decays if its wave-number is sufficiently large, $|k| > k_0$ with $k_0 = \cos^{-1}(D/2)$.

Uniform Distribution. Since $k_0 = 0$ for $D = 2$, there is a critical diffusivity

$$D_c = 2. \tag{11}$$

For strong diffusion, $D > D_c$, perturbations decay exponentially with time and the uniform state is rapidly restored, regardless of the initial conditions (figures 3a-3c). Just above the critical diffusivity, long wavelength perturbations are long lived: their decay time diverges $\propto (D - D_c)^{-2}$, following from $\lambda \propto (D - D_c)\, k^2$ as $D \downarrow D_c$. In any case, compromise interactions become irrelevant and diffusion dominates. As a result, the opinion distribution approaches a structureless state: the political system is disorganized as no parties are formed.

Coarsening. For weak diffusion, $D < D_c$, perturbations to the initial state are magnified and parties are quickly formed (figure 3d-3f). The system develops a mosaic of parties. Since the initial state is heterogeneous, the size of the parties and the separation between them vary. The evolution follows straightforwardly from the two-party dynamics and there is a linear profile in regions separating parties. Small parties merge into larger neighboring parties, and as a result, the remaining parties grow in size and in niche (figure 4).

Let us assume that the typical size is m. The population density must be constant, so the typical niche is

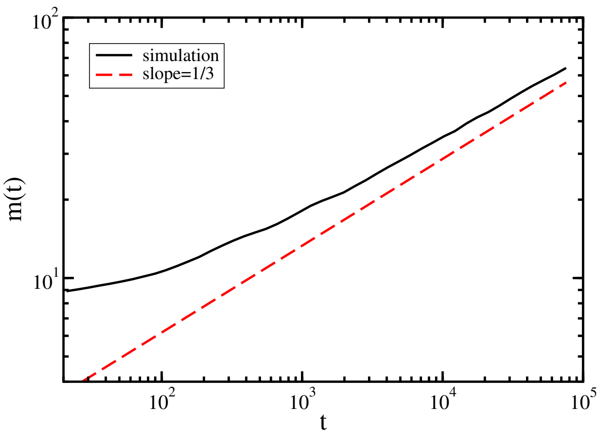

FIG. 5: The average party size versus time. Shown are simulation results (solid line) obtained using $D = 1$ and $N = 10^5$ and a line of slope $= 1/3$ (dashed line) for reference.

of the same order, $l \sim m$. Substituting these scales into the depletion time (9) yields $T \sim m^3$ and since time is the only relevant time scale, the typical size growth law is (figure 5)

$$m \sim t^{1/3}. \tag{12}$$

Asymptotically, the system reaches a self-similar state where the party size is characterized by the typical scale m and consequently, the party size distribution, Q_m, becomes self-similar in the long time limit, $Q_m \sim t^{-1/3}\Psi(mt^{-1/3})$. Our numerical simulations confirm this, along with (12).

The lifetime of a party is governed by the interplay between size and niche. Typically, the larger the party, the longer it survives, but a small party may still outlast a larger neighbor if it has a large enough niche. Except for this size-niche competition, the coarsening mechanism is similar to Lifshitz-Slyozov ripening [20].

Pattern Formation. Without diffusion, the system approaches a state where $P_{n-1}P_{n+1} = 0$ for all n; there is no evolution and parties are localized to either one or two sites [16]. For a narrow political spectrum, consensus is reached and there is a single party. As the size of the spectrum increases, the number of parties undergoes a series of bifurcations and there may be two off-center parties, three parties, etc [11, 16].

Asymptotically, the number of parties grows linearly with the political spectrum N because the parties are equally-spaced (figure 3i). The spacing equals the party size, as follows from conservation of population. This pattern formation can be understood using linear stability analysis. We demonstrate this for uniform initial distributions.

Consider the uniform initial condition: $P_n(0) = 0$ for $n < 0$ and $P_n(0) = 1$ for $n \geq 0$. This state is unstable with respect to perturbations that propagate from the boundary into the unstable uniform state (figures 3g-3i). A small periodic disturbance $P_n = 1 + \phi_n$ with $\phi_n \propto \exp[i(kn - \omega t)]$ is characterized by the dispersion relation $\omega = 2i(2\cos k - \cos 2k - 1)$ according to the diffusion-free evolution equation (3). A saddle point analysis shows that the propagation velocity v obeys [21]

$$v = \frac{d\omega}{dk} = \frac{\mathrm{Im}[w]}{\mathrm{Im}[k]}. \tag{13}$$

The solution is $k = k_{\rm select} + i\lambda$ with the selected wave number $k_{\rm select} = 1.183032$. The decay constant $\lambda = 0.467227$ characterizes the exponential decay far into the unstable state, $\phi_n \sim \exp[-\lambda(n - vt)]$. The propagation velocity is $v = 3.807397$ and the period of the pattern is $L_{\rm select} = 2\pi/k_{\rm select} = 5.311086$. Our numerical studies confirm these results.

In the absence of boundaries, i.e., in a periodic system, the perturbation with the largest growth rate dominates and sets the wavelength. From (10), the growth rate is $\lambda = 2(2\cos k - \cos 2k - 1)$; it is periodic in k, $\lambda(k) = \lambda(k + 2\pi)$. The uniform state is unstable with respect to long wavelength perturbations with $k < \pi/2$. Also, the growth rate is maximal at $k_{\rm max} = \pi/3$ and the corresponding period, $L_{\rm max} = 2\pi/k_{\rm max}$, is $L_{\rm max} = 6$.

Numerically, we find that the actual period falls between the two linear stability values

$$L \approx 5.67. \tag{14}$$

Starting from a compact distribution, perturbations with the selected period are generated by the boundary and they propagate into the interior. As the disturbance reaches the interior of the system, perturbations with a smaller wavelength, that have a larger growth rate, dominate. This argument suggests the above estimates as bounds for the period $L_{\rm select} < L < L_{\rm max}$. These bounds are tight, so linear stability analysis yields a very good approximation for the period. Yet, the pattern selection mechanism is intrinsically nonlinear and obtaining the exact period remains a challenge.

In summary, we found that the level of noise (diffusion) determines the nature of the political system. Strong noise leads to a uniform distribution of opinions with every possible opinion equal in weight. With weak noise, the system organizes into political parties and the political landscape undergoes coarsening with large parties continuously overtaking small ones. Without noise, the system evolves into a frozen pattern of parties with equal weights and equal separations.

Several qualitative features are surprisingly realistic. The lifecycle of a party includes formation, growth, and demise. Isolated parties have a fixed position and their lifetime grows rapidly with their size, but ultimately, any party dissolves. When two parties interact, the smaller party loses ground to the larger party at a steady rate. A small party with a large niche can be long lived.

Diffusion plays a critical role. It facilitates interaction between parties and it is responsible for the dissolution of parties. Moreover, the patterned state is unstable with respect to addition of diffusion. We conclude that spontaneous opinion changes are an integral part of opinion dynamics.

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[1] W. Weidlich, *Sociodynamics: A Systematic Approach to Mathematical Modelling in the Social Sciences* (Harwood Academic Publishers, 2000).
[2] T. M. Liggett, *Interacting particle systems* (Springer-Verlag, New York, 1985).
[3] S. Galam, Y. Gefen, and Y. Shapir, Math. J. Soc. **9**, 1 (1982); S. Galam and S. Moscovivi, Eur. J. of Soc. Psych. **21**, 49 (1992).
[4] R. Axelrod, *The complexity of cooperation* (Princeton University Press, 1997).
[5] S. Galam, Physica A **333**, 453 (2004).
[6] E. Ben-Naim, L. Frachebourg, and P. L. Krapivsky, Phys. Rev. E **53**, 3078 (1996).
[7] K. Sznajd-Weron and J. Sznajd, Int. J. Mod. Phys. C **11**, 1157 (2000).
[8] D. Stauffer, J. Artif. Soc. Soc. Simul. **5**, 477 (2002).
[9] A. Soulier and T. Halpin-Healy, Phys. Rev. Lett. **90**, 258103 (2003).
[10] M. Mobilia and S. Redner, Phys. Rev. E **68**, 046106 (2003).
[11] G. Deffuant, D. Neau, F. Amblard and G. Weisbuch, Adv. Comp. Sys. **3**, 87 (2000).
[12] G. Weisbuch, G. Deffuant, F. Amblard, and J. P. Nadal, Complexity **7**, 55 (2002).
[13] R. Hegselman and U. Krause, J. Art. Soc. Soc. Sim. **5**, 2 (2002).
[14] D. Stauffer and H. Meyer-Ortmanns, Int. J. Mod. Phys. B **15**, 241 (2004).
[15] S. Fortunato, cond-mat/0406054.
[16] E. Ben-Naim, P. L. Krapivsky, and S. Redner, Physica D **183**, 190 (2003)
[17] E. Ben-Naim, P. L. Krapivsky, R. Vazquez, and S. Redner, Physica A **330**, 99 (2003).
[18] For $P_n(0) = m\delta_{n,0}$ the core has a different shape $(P_0, P_1, P_2, P_3) = (m, D^{1/2}m^{-1/2}, D^{3/2}m^{1/2}, D^2m^{-1})$, but essential features (height and depth) are the same.
[19] H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, (Oxford University Press, New York 1959).
[20] I. M. Lifshitz and V. V. Slyozov, Zh. Eksp. Teor. Fiz. **35**, 479 (1959) [Sov. Phys. JETP **8**, 331 (1959)].
[21] W. van Saarloos, Phys. Rep. **386**, 29 (2003).