Nonlinear Integral-Equation Formulation of Orthogonal Polynomials

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Abstract. The nonlinear integral equation $P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P(x+y)$ is investigated. It is shown that for a given function $w(x)$ the equation admits an infinite set of polynomial solutions $P(x)$. For polynomial solutions, this nonlinear integral equation reduces to a finite set of coupled linear algebraic equations for the coefficients of the polynomials. Interestingly, the set of polynomial solutions is orthogonal with respect to the measure $x\, w(x)$. The nonlinear integral equation can be used to specify all orthogonal polynomials in a simple and compact way. This integral equation provides a natural vehicle for extending the theory of orthogonal polynomials into the complex domain. Generalizations of the integral equation are discussed.

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There are many ways to specify uniquely a set of orthogonal polynomials. One can specify the domain (α, β) and the measure with respect to which the polynomials are orthogonal and then use the cumbersome Gramm-Schmidt orthogonalization procedure to construct the polynomials. For example, for the domain $(-1, 1)$ and measure $(1-x^2)^{-1/2}$, the Gramm-Schmidt procedure yields the Chebyshev polynomials $T_n(x)$. Alternatively, one can specify a recursion relation. The linear recursion relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ along with the initial conditions $T_0(x) = 1$ and $T_1(x) = x$ again produces the Chebyshev polynomials. Another approach is to give the differential-equation eigenvalue problem satisfied by the polynomials. The Chebyshev polynomials $T_n(x)$ satisfy the eigenvalue equation $(1-x^2)y''(x) - xy'(x) + n^2 y(x) = 0$. Stating the generating function is yet another way to specify a set of polynomials. For the Chebyshev polynomials the generating function

$$\frac{1-xt}{1-2xt+t^2} = \sum_0^\infty T_n(x) t^n$$

uniquely defines these polynomials.

In this paper we propose a simple and compact way to specify a set of orthogonal polynomials in terms of a nonlinear integral equation. Consider the nonlinear integral

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equation

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P(x+y). \tag{1}$$

The integration limits α and β as well as the function $w(x)$ are arbitrary except for the restriction that $w(x)$ must have a nonvanishing integral, $\int_\alpha^\beta dx\, w(x) \neq 0$. Therefore, without loss of generality, we may assume that w is normalized:

$$\int_\alpha^\beta dx\, w(x) = 1.$$

Note that if we choose $\alpha = -\infty$, $\beta = \infty$, and $w(y) = e^{iy}$, (1) reduces to the heavily studied equation for the Wigner function [1, 2]. The original motivation for considering this nonlinear integral equation is that it describes a stochastic process in which two random variables are subtracted to create a new one. The steady-state probability distribution for this random variable satisfies the integral equation (1) with $\alpha = 0$, $\beta = \infty$, and $w(y) = 2$.

Even though this integral equation is nonlinear, its polynomial solutions can be found analytically. By inspection one can see that there is the trivial solution $P_0(x) = 1$. However, there are also infinitely many other polynomial solutions because (1) has two remarkable properties.

First, if we seek a solution in the form of an arbitrary polynomial of degree n,

$$P_n(x) = \sum_{k=0}^{n} a_{n,k}\, x^k, \tag{2}$$

the nonlinear integral (1) preserves the degree of the polynomial. For example, if we substitute an arbitrary linear polynomial $P_1(x) = a_{1,0} + a_{1,1}x$ or an arbitrary quadratic polynomial $P_2(x) = a_{2,0} + a_{2,1}x + a_{2,2}x^2$ into (1), we obtain

$$a_{1,0} + a_{1,1}x = \langle P_1(y)\, [a_{1,0} + a_{1,1}y + a_{1,1}x] \rangle$$

and

$$a_{2,0} + a_{2,1}x + a_{2,2}x^2 = \langle P_2(y) \left[a_{2,0} + a_{2,1}y + a_{2,1}x + a_{2,2}y^2 + 2a_{2,2}xy + a_{2,2}y^2\right] \rangle,$$

where we have introduced the notation

$$\langle f \rangle \equiv \int_\alpha^\beta dx\, w(x) f(x). \tag{3}$$

In general, for an arbitrary polynomial of degree n both the left and right sides of (1) are polynomials of the same degree n.

Substituting a polynomial of degree n into the right side of (1) and performing the integration, we obtain a polynomial of degree n.

Second, for the case of polynomial solutions, the nonlinear equation (1) reduces to a system of coupled *linear* equations, which are obtained by equating like powers of x.

For the case $n = 1$ above, equating the coefficient of x^1 on both sides of the equation gives $a_{1,1} = a_{1,1}\langle P_1(y)\rangle$. We require that the polynomial have degree one, $a_{1,1} \neq 0$, so

$$\langle P_1(x)\rangle = 1, \tag{4}$$

where we have replaced y by x. Next, we equate the coefficients of x^0 and get $a_{1,0} = a_{1,0}\langle P_1(y)\rangle + a_{1,1}\langle y\, P_1(y)\rangle$. Substituting (4) into this equation, we obtain

$$\langle x\, P_1(x)\rangle = 0. \tag{5}$$

Thus, the nonlinear equation (1) reduces to a system of two coupled inhomogeneous linear equations for the two unknowns $a_{1,0}$ and $a_{1,1}$. The integral equation reduces to a linear algebraic system only for polynomials because, as we see in the above calculation, the derivation relies on the fact that the power series in x truncates at a finite order.

Let us repeat this procedure for quadratic polynomials: Starting at the highest power x^2, we find that $a_{2,2} = a_{2,2}\langle P_2(y)\rangle$ and hence we recover (4) with the subscript 1 replaced by 2. Setting the coefficients of x^1 equal gives $a_{2,1} = a_{2,1}\langle P_2(y)\rangle + 2a_{2,2}\langle y\, P_2(y)\rangle$ and we recover (5) with the subscript 1 replaced by 2. The coefficient of x^0 gives $a_{2,0} = a_{2,0}\langle P_2(y)\rangle + a_{2,1}\langle y\, P_2(y)\rangle + a_{2,2}\langle y^2\, P_2(y)\rangle$. Therefore, there is now the third equation

$$\langle x^2\, P_2(x)\rangle = 0. \tag{6}$$

The general pattern is clear. An nth-degree polynomial is a solution of the integral equation if and only if the following set of $n + 1$ linear equations is satisfied:

$$\langle x^k P_n(x)\rangle = \delta_{k,0} \qquad (k = 0, 1, \ldots, n). \tag{7}$$

The linear equations (7) imply that the polynomials $P_n(x)$ are orthogonal with respect to the measure

$$g(x) = x\, w(x). \tag{8}$$

If $P_m(x) = \sum_{k=0}^{m} a_{m,k}\, x^k$ is a polynomial of degree $m < n$, then from (7) we conclude that

$$\langle x\, P_n P_m\rangle = \sum_{k=0}^{m} a_{m,k}\langle x^{k+1}\, P_n(x)\rangle = \sum_{k=1}^{m+1} a_{m,k-1}\langle x^k\, P_n(x)\rangle = 0. \tag{9}$$

This is the main result of our paper. We have shown that the nonlinear integral equation (1) admits an infinite set of polynomial solutions. This set is unique; there is one and only one polynomial solution of degree n. These polynomials are orthogonal.

We comment that this integral formulation of orthogonal polynomials is general. The integration limits and the function $w(x)$ are arbitrary apart from the restriction that $w(x)$ have a positive integral $\int_\alpha^\beta dx\, w(x) > 0$.

We mention three classical orthogonal polynomials [3] that can be generated using this approach:

- Generalized Laguerre polynomials $L_n^{(\gamma)}(x)$ using $\alpha = 0$, $\beta = \infty$, and $w(x) = x^{\gamma-1}e^{-x}$ for all $\gamma \geq 1$;

• Jacobi polynomials $G_n(p,q,x)$ using $\alpha = 0$, $\beta = 1$, and $w(x) = x^{q-2}(1-x)^{p-q}$, for $q > 1$ and $p - q > -1$;

• Shifted Chebyshev polynomials of the second kind $U_n^*(x)$ using $\alpha = 0$, $\beta = 1$, and $w(x) = (1-x)^{1/2}x^{-1/2}$.

The equations (7) can be written compactly as $n+1$ simultaneous linear equations for the $n+1$ coefficients $a_{n,j}$:

$$\sum_{j=0}^{n} a_{n,j} m_{k+j} = \delta_{k,0} \qquad (k = 0, 1, \ldots, n). \tag{10}$$

Here $m_n = \langle x^n \rangle$ are the moments of $w(x)$ and by assumption $m_0 = 1$. Thus, using Cramer's rule we can express the polynomial solutions explicitly as ratios of determinants [4]. We define the matrices

$$A_n = \begin{pmatrix} 1 & x & \cdots & x^n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{m+1} & \cdots & m_{2n} \end{pmatrix}, \quad B_n = \begin{pmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{m+1} & \cdots & m_{2n} \end{pmatrix},$$

so that $B_n = \langle A_n \rangle$. The polynomials $P_n(x)$ are then given by

$$P_n(x) = \frac{\det A_n}{\det B_n}. \tag{11}$$

The normalization of the polynomials can be given in the form

$$\langle x\, P_n(x)\, P_m(x) \rangle = \delta_{n,m}\, G_n. \tag{12}$$

The normalization factors G_n are ratios of determinants of matrices

$$G_n = \frac{(\det C_n)(\det C_{n+1})}{(\det B_{n+1})^2}, \tag{13}$$

where the matrix C_n is given by

$$C_n = \begin{pmatrix} m_1 & m_2 & \cdots & m_n \\ m_2 & m_3 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{m+1} & \cdots & m_{2n} \end{pmatrix}.$$

For completeness, we give the first three polynomials explicitly:

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= \frac{m_2 - x m_1}{m_2 - m_1^2}, \\ P_2(x) &= \frac{(m_2 m_4 - m_3^2) + (m_2 m_3 - m_1 m_4)x + (m_1 m_3 - m_2^2)x^2}{m_4(m_2 - m_1^2) - m_3^2 + 2 m_1 m_2 m_3 - m_2^3}. \end{aligned}$$

Also, we give the corresponding normalization factors G_n:

$$
\begin{aligned}
G_0 &= m_1,\\
G_1 &= m_1\frac{m_1m_3-m_2^2}{(m_2-m_1^2)^2},\\
G_2 &= \frac{(m_1m_3-m_2^2)(m_1m_3m_5-m_2^2m_5-m_1m_4^2+2m_2m_3m_4-m_3^3)}{[m_4(m_2-m_1^2)-m_3^2+2m_1m_2m_3-m_2^3]^2}.
\end{aligned}
$$

Note that (11–13) are only valid if $\det B_n \neq 0$. This condition holds when the measure $g(x) = x\,w(x)$ is positive on $\alpha \le x \le \beta$ [5, 6].

The integral equation (1) specifies all possible orthogonal polynomials. When the weight function $g(x)$ with respect to which the polynomials are orthogonal does not vanish at $x = 0$ and consequently $w(x) = g(x)/x$ is singular at $x = 0$, the path of integration from α to β may be taken in the complex plane to avoid the singularity at the origin. Using a complex integration path, all steps leading to (9) are valid.

For example, consider the Legendre polynomials for which $\alpha = -1$, $\beta = 1$, and $g(x)$ is a constant. There are an infinite number of topologically distinct integration paths that connect -1 to 1. These paths are characterized by their winding numbers. For definiteness, we choose a path that goes from -1 to 1 in the positive (counterclockwise) direction and does not encircle the origin. On this path $\int dx/x = i\pi$ and hence to maintain the normalization $\int_\alpha^\beta dx\, w(x) = 1$ we use $w(x) = 1/(i\pi x)$. The moments $m_n = \langle x^n\rangle$ are $m_0 = 1$, $m_1 = 2/(i\pi)$, $m_2 = 0$, $m_3 = 2/(3i\pi)$, $\ldots$. From the moment formulas (11), we obtain

$$
\begin{aligned}
P_0(x) &= 1,\\
P_1(x) &= \frac{i\pi}{2}x,\\
P_2(x) &= 1-3x^2,\\
P_3(x) &= \frac{3i\pi}{8}(3x-5x^3),
\end{aligned}
\tag{14}
$$

and so on. These polynomials are the standard Legendre polynomials, except that the odd polynomials have an imaginary multiplicative factor that is determined by the winding number of the integration path. Of course, these polynomials are solutions of the linear equations (7).

To summarize, while the usual theory of orthogonal polynomials is formulated in terms of real integrals, the integral equation (1) provides a simple and natural framework to extend the theory of orthogonal polynomials into the complex domain. In doing so we discover an interesting connection between the polynomial coefficients and the topological winding number of the integration path.

There are many ways to generalize the integral equation (1):

1. Multiplicative argument. If we replace the term $P(x+y)$ in the original nonlinear integral equation (1) by $P(xy)$, we obtain a new class of nonlinear equations:

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P(x\,y). \tag{15}$$

Each of these nonlinear integral equations also has an infinite number of polynomial solutions.

Again, there is the constant solution $P_0(x) = 1$. To find other solutions we substitute a polynomial of degree n and then equate coefficients of x^k on the left and right sides, starting with $k = n$. The nonlinear integral equation reduces to a set of $n+1$ inhomogeneous equations of the form

$$a_{n,k}\langle x^k P_n(x)\rangle = a_{n,k} \qquad (k = 0, 1, \ldots, n). \tag{16}$$

However, unlike the previous case, the equations are quadratic and there are now 2^{n-1} solutions because each of the coefficients $a_{n,k}$ can be either zero or nonzero for $k = 0, \ldots, n-1$. For each nonzero coefficient the linear equation

$$\langle x^k P_n(x)\rangle = 1 \tag{17}$$

holds for all k for which $a_{n,k} \neq 0$.

In one special class of solutions all of the coefficients are nonzero and the polynomials are orthogonal with respect to the measure

$$g(x) = (1-x)\,w(x). \tag{18}$$

To verify this, we take a polynomial of degree $m < n$ and observe that

$$\langle (1-x)P_n P_m\rangle = \sum_{k=0}^{m} a_{m,k}\left(\langle x^k P_n\rangle - \langle x^{k+1} P_n\rangle\right) = 0.$$

This equation is valid because all of the terms in the parentheses vanish by virtue of (17). This measure is relevant for the class of polynomials for which $0 \leq \alpha < \beta \leq 1$.

Here are two classical orthogonal polynomials that can be generated in this way:

• Jacobi polynomials $G_n(p, q, x)$ using $\alpha = 0$, $\beta = 1$, and $w(x) = (1-x)^{p-q-1}x^{q-1}$ with $p - q > 0$ and $q > 0$;

• Shifted Chebyshev polynomials of the second kind $U_n^*(x)$ using $\alpha = 0$, $\beta = 1$, and $w(x) = x^{1/2}(1-x)^{-1/2}$.

In another class of solutions, the coefficients alternate between zero and nonzero so that the polynomials alternate between definite even and odd parity. The polynomials are orthogonal with respect to a measure whose moments μ_n are

$$\mu_n = \tfrac{1}{2}\langle x^n - x^{n+2}\rangle\left[1 + (-1)^n\right]. \tag{19}$$

Thus, all the even moments of the measure are positive and all the odd moments vanish. It is easy to show that polynomials of similar parity are orthogonal with respect to the measure $(1-x^2)\,w(x)$. Polynomials of dissimilar parity are orthogonal because their product is an odd polynomial, and the odd moments vanish (19).

2. Linear arguments. For the integral equation with a linearly shifted argument,

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P(x + a + b\,y), \tag{20}$$

where $b \neq 0$ is an arbitrary constant, polynomials of degree n are solutions when $\langle (a+bx)^k\, P_n(x)\rangle = \delta_{k,0}$. This implies that the polynomials are orthogonal with respect to the measure $g(x) = (a + bx)\, w(x)$. Note that when $a = 0$, the polynomials are identical to those generated by the original integral formula (1). Curiously, for $a = 1$ and $b = -1$ we recover the polynomials generated by the integral equation (15).

3. Functional arguments. The integral equation

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P[x + f(y)], \tag{21}$$

where $f(x)$ is a nonconstant function, has polynomial solutions of degree n when $\langle [f(x)]^k\, P_n(x)\rangle = \delta_{k,0}$. These polynomials are not necessarily an orthogonal set. Nevertheless, the polynomial $P_n(x)$ is orthogonal to the function $P_m[f(x)]$ when $m < n$ with respect to the measure $g(x) = f(x)\, w(x)$.

4. Arbitrary functions. The integral equation

$$P(x) = \int_\alpha^\beta dy\, w(y)\, f[P(y)]\, P(x + y), \tag{22}$$

where $f(x)$ is a nonconstant function, has polynomial solutions of degree n when $\langle x^k f[P_n(x)]\rangle = \delta_{k,0}$. This implies that the function $f[P_n(x)]$ is orthogonal to the polynomial $P_m(x)$ with respect to the measure $g(x) = x\, w(x)$.

5. Further generalizations. It is worth considering what happens when the function $w(x)$ is singular on the interval $\alpha < x < \beta$. Also, an obvious way to generalize (1) is to iterate it, and thereby to obtain integral equations that are cubic, quartic, and so on. Furthermore, one can study the properties of nonpolynomial solutions to (1). We have found many such solutions. Finally, one can generalize (1) to multidimensional integrals and study the properties of the resulting multivariate polynomial solutions.

In summary, we have shown that all sets of orthogonal polynomials are solutions of nonlinear integral equations. For polynomial solutions these nonlinear equations reduce to simultaneous linear equations for the coefficients of the polynomials. The measure with respect to which the polynomials are orthogonal depends on the form of the integral equation and on the integration measure. The nonlinear integral equations discussed here provide a simple and compact way to define a set of orthogonal polynomials and also provide a framework for extending the general theory of orthogonal polynomials into the complex domain.

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