Kinetic Theory of Random Graphs

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Abstract.
Statistical properties of evolving random graphs are analyzed using kinetic theory. Treating the linking process dynamically, structural characteristics of links, paths, cycles, and components are obtained analytically using the rate equation approach. Scaling laws for finite systems are derived using extreme statistics and scaling arguments.

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INTRODUCTION

Random graphs have been studied in many disciplines including statistical physics, chemical physics, combinatorics, probability theory, and computer science [1, 2, 3, 4, 5, 6]. For example, they are used to model percolation in polymerization processes [7, 8] and phase transitions in algorithmic complexity [9].

A random graph is a set of nodes that are joined by random links. When the number of links exceeds a threshold, a connected component containing a finite fraction of all nodes, the giant component, emerges. Essentially, random graphs are a mean-field model of percolation [10, 11].

In this short review, we summarize our recent work on random graphs [12, 13]. We describe how, by treating the linking process dynamically, random graphs can be studied using kinetic theory. Structures such as paths, cycles, and components grow via elementary aggregation processes and their distributions can be obtained using the rate equation approach [14, 15, 16, 17, 18, 19]. This technique complements the combinatorial methods, traditionally used to analyze random graphs [3, 4, 5].

THE EVOLVING RANDOM GRAPH

A graph is a collection of nodes that are joined by links, and in a random graph, the links are random. There are different types of graphs. In a static graph, the links are generated instantaneously, while in an evolving graph the links are generated sequentially. In a simple graph, a given pair of nodes may be connected by a single link only, but in a multigraph they may be connected by multiple links.

We consider the following random graph model. Starting with no links and N disconnected nodes, links are sequentially added between randomly selected pairs of nodes. This linking process continues ad infinitum with constant rate, set to $(2N)^{-1}$ without loss of generality. The two nodes selected for linking may not be different, and additionally, the number of links between two nodes is not limited. In other words, we consider a random evolving multi-graph. Additionally, we consider the infinite system size limit, $N \to \infty$, where statistical fluctuations can be usually ignored.

LINKS, PATHS, AND CYCLES

At time t, the total number of links is on average $Nt/2$, and therefore, the average number of links per node (the degree) equals time t. Let l be the degree of a node. It undergoes the additive stochastic process $l \to l+1$. The probability F_l that the degree of a node equals l, the degree distribution, satisfies

$$\frac{dF_l}{dt} = F_{l-1} - F_l \tag{1}$$

with the initial condition $F_l(0) = \delta_{l,0}$. Therefore, the degree distribution is Poissonian

$$F_l = \frac{t^l}{l!} e^{-lt} \tag{2}$$

with the mean degree equal to time, $\langle l \rangle = t$.

A pair of nodes may be connected by a consecutive series of links forming a path. When a newly added link connects two paths of lengths n and m, a longer path is formed: $n, m \to n+m+1$. Thus, paths undergo an aggregation process. Let $P_l(t)$ be the density of *distinct* paths containing l links at time t. This density satisfies

$$\frac{dP_l}{dt} = \sum_{n+m=l-1} P_n P_m \tag{3}$$

for $l > 0$ and $P_0(t) = 1$. The initial condition is $P_l(0) = \delta_{l,0}$. Therefore, the path length density is

$$P_l = t^l. \tag{4}$$

For example, the first quantity $P_1 = t$ reflects that the link density is equal to $t/2$ and that every link corresponds to two distinct paths of length one. The total density of paths $P_{\rm tot} \equiv \sum_l P_l$ diverges at the percolation time, $P_{\rm tot} = (1-t)^{-1}$ as $t \to 1$. At this time, the system develops a giant component that eventually percolates through the entire system. The divergence of the total number of paths is typical: average quantities as well as typical characteristics diverge near the transition point. For example, the typical path length diverges as the percolation time is approached

$$l \sim (1-t)^{-1}, \tag{5}$$

as seen from the average path length, $\langle l \rangle = \sum_l l P_l / \sum_l P_l = t(1-t)^{-1}$.

When two nodes along a path are linked, a cycle forms. Cycles have been studied extensively [20, 21, 22, 23] and for example, they are useful for characterizing phase transitions in algorithmic complexity [9]. Let the average number of cycles of size l at time t be $Q_l(t)$. It is coupled to the path length density via the rate equation

$$\frac{dQ_l}{dt} = \frac{1}{2} P_{l-1}. \tag{6}$$

The right-hand side equals the link creation rate $1/(2N)$ times the total number of paths NP_{l-1}. As a result, the cycle length distribution is

$$Q_l = \frac{t^l}{2l}. \tag{7}$$

Thus, at the percolation time, the cycle length distribution is inversely proportional to the length, $Q_l(t=1) = (2l)^{-1}$. In general, size distributions decay exponentially away from the percolation point and algebraically precisely at the percolation point. The total number of cycles in the system $Q_{\text{tot}} \equiv \sum_l Q_l$ is $Q_{\text{tot}} = \frac{1}{2} \ln \frac{1}{1-t}$. It weakly diverges as the percolation point is approached. We note that the average number of cycles is not an extensive quantity and for large systems, it saturates at a finite value. The number of cycles is therefore a non-self-averaging quantity, it fluctuates from realization to realization.

What is the probability $S(t)$ that the system contains no cycles at time t? Since the cycle formation process is random, and the cycle production rate is $J = dQ_{\text{tot}}/dt = \frac{1}{2(1-t)}$, then $dS/dt = -SJ$ or alternatively,

$$\frac{dS}{dt} = -\frac{S}{2(1-t)}. \tag{8}$$

Therefore, the probability that the system contains no cycles decays with time as follows

$$S = (1-t)^{1/2}. \tag{9}$$

This survival probability shows that the system is bound to nucleate at least a single cycle prior to the percolation time.

Following similar reasoning, properties of the first cycle may be obtained. The size distribution of the first cycle G_l obeys a simple generalization of (6)

$$\frac{dG_l}{dt} = \frac{1}{2} S P_{l-1}. \tag{10}$$

The rate by which the first cycle is produced is simply the rate by which all cycles are produced times the probability that there are no cycles in the system. Since the first cycle must be produced by the percolation time, the final size distribution of the first cycle is $G_l(1) = \frac{1}{2} \int_0^1 dt\, S P_{l-1}$ and performing the integration gives

$$G_l(t=1) = \frac{\Gamma(3/2)\Gamma(l)}{2\Gamma(l+3/2)}. \tag{11}$$

This size distribution has an algebraic tail, $G_l(1) \sim l^{-3/2}$ for $l \gg 1$. The characteristic exponent for the tail of G_l is larger than the characteristic exponent for the tail of Q_l: the first cycle is created earlier, and thus, it must be smaller.

FINITE COMPONENTS: SIZE DISTRIBUTION

A component is a set of connected nodes: every pair of nodes in a component is connected by a path. Components merge due to linking: a link placed between two distinct components causes the two to join. There are $i \times j$ ways to join disconnected components of size i and j and hence, components undergo the aggregation process $(i, j) \to i + j$ with the aggregation rate $ij/(2N)$.

Let $c_k(t)$ be the density of components containing k nodes at time t. The component size distribution obeys the nonlinear rate equation

$$\frac{dc_k}{dt} = \frac{1}{2} \sum_{i+j=k} (ic_i)(jc_j) - k\, c_k. \tag{12}$$

The initial condition is $c_k(0) = \delta_{k,1}$. The gain term represents merger between two components whose sizes sum up to k and the loss term accounts for links involving a node inside a component of size k.

The moments of the size distribution, $M_n = \sum_k k^n c_k$, provide a useful probe of the dynamics. For example, the second moment obeys the closed equation $dM_2/dt = M_2^2$ and with the initial condition $M_2(0) = 1$, the solution is

$$M_2 = (1-t)^{-1}, \tag{13}$$

for $t < 1$. The divergence shows that the system undergoes a percolation transition. In a finite time $t_g = 1$, an infinite component, the giant component, is formed. Past the percolation point, the giant component contains a finite fraction of the nodes, and eventually, it grows to engulf the entire system.

The component size distribution can be obtained analytically[1]

$$c_k(t) = \frac{k^{k-2}}{k!}\, t^{k-1}\, e^{-kt}. \tag{14}$$

This size distribution decays exponentially away from the percolation point and algebraically at the percolation point, $c_k(1) \simeq (2\pi)^{-1/2} k^{-5/2}$. Both behaviors follow from the scaling behavior $c_k(t) \to (1-t)^5 \Phi_C\big(k(1-t)^2\big)$ with the typical component size

$$k \sim (1-t)^{-2} \tag{15}$$

and the scaling function $\Phi_C(x) = (2\pi)^{-1/2} x^{-5/2} \exp(-x/2)$. The large-size algebraic decay of the size distribution is reflected by the small-argument behavior of the scaling

[1] A convenient solution method is as follows. The time dependence is "peeled" first, $c_k = C_k t^{k-1} e^{-kt}$ with the coefficients satisfying $(k-1)C_k = \sum_{i+j=k} (iC_i)(jC_j)$. The generating function $G(z) = \sum_k kC_k e^{kz}$ satisfies the differential equation $(1-G)G' = G$ and consequently $Ge^{-G} = e^z$. The coefficients are found using the Lagrange inversion formula [13]. The very same technique can be used to derive the joint distributions described in the next section.

function. Hence, the size distribution exhibits dynamical scaling in the vicinity of the percolation transition. This behavior is generic: size distributions obey dynamical scaling near the percolation point and for example, both the path length density (4) and the cycle length density (7) can be written in a self-similar form.

Other statistical properties including for example the moments follow from the generating function, $c(z,t)=\sum_k k\,c_k(t)e^{kz}$, that can be written explicitly

$$c(z,t)=t^{-1}G(z+\ln t-t), \tag{16}$$

in terms of the auxiliary generating function

$$G(z)=\sum_{k=1}^{\infty}\frac{k^{k-1}}{k!}e^{kz}. \tag{17}$$

This function satisfies $Ge^{-G}=e^z$ or alternatively, $(1-G)dG/dz=G$.

Combining the latter relation and the behavior of the generating function near $z=0$, the second moment result (13) is generalized to all times, $M_2=u/[t(1-u)]$ with $u=G(\ln t-t)$. This quantity satisfies

$$u\,e^{-u}=t\,e^{-t}. \tag{18}$$

Let the fraction of nodes outside finite components be $g=1-M_1$ with $M_1=c(z=0)=u/t$. For $t<1$, there is a single solution $u=t$ and therefore all nodes are in finite components, $g=0$. But when $t>1$, there is an additional nontrivial solution, and as a result, this fraction becomes finite, $g>0$. In particular, at the late stages of the process, the giant component contains almost all nodes, $g(t)\to 1$, and furthermore, since $u\simeq te^{-t}$ then $1-g\simeq c_1=e^{-t}$, indicating that other than the giant component, there are only a few isolated nodes. Also, just past the percolation point, the fraction of nodes outside finite components grows linearly, $g(t)\simeq 2(t-1)$.

FINITE COMPONENTS: JOINT DISTRIBUTIONS

We have seen that components undergo an aggregation process of one kind and that links, paths, and cycles undergo growth or aggregation processes of another kind. By combining these separate processes into a bi-aggregation process [24] involving two variables, say the component size and the node degree, a more detailed analysis of structural properties of finite components is possible.

Links

Each node can be characterized by two indices: its degree l and the size k of the component it belongs to. The distribution $f_{l,k}$ of nodes of degree l in components of size k satisfies

$$\frac{df_{l,k}}{dt}=\sum_{i+j=k}(jc_j)[(i-1)f_{l,i}+f_{l-1,i}]-k\,f_{l,k}. \tag{19}$$

The initial conditions are $f_{l,k}(0) = \delta_{l,0}\,\delta_{k,1}$. The first gain term accounts for linking events that leave the node degree unchanged (the added link involves other nodes in the component), while the second gain term represents linking events that augment the node degree by one. Of course, the component size distribution and the node-degree distribution can be obtained by summation of the joint distribution, so that Eqs. (1) and (12) can be recovered from (19).

The joint distribution can be obtained analytically

$$f_{l,k}(t) = \frac{(k-1)^{k-l-2}}{(l-1)!(k-l-1)!}\,t^{k-1}\,e^{-kt} \tag{20}$$

for $1 \le l < k$ and $f_{0,k}(t) = \delta_{k,1}\,e^{-t}$ for $l = 0$. Fixing the node degree, the joint distribution decays algebraically at large sizes at the critical point $f_{l,k}(t=1) \sim k^{-3/2}$. Exponential decay occurs elsewhere.

The generating function $f(z,w) = \sum_{l,k} e^{kz} w^l f_{l,k}$ is expressed directly in terms of auxiliary generating function

$$f(z,w) = e^{wG(z+\ln t-t)+z-t}. \tag{21}$$

Average quantities and correlations follow from the generating function, and for example, the average node degree, the average component size, and the average correlation between the two are

$$\langle l\rangle = u, \qquad \langle k\rangle = \frac{1}{1-u}, \qquad \langle kl\rangle = \frac{2u}{1-u}. \tag{22}$$

Below the percolation transition, the average node degree equals time, $\langle l\rangle = t$. Above the percolation transition, the average node degree is reduced $\langle l\rangle < t$ because nodes in finite components have less connections than the rest of the nodes. The average degree vanishes in the long time limit $\langle l\rangle \simeq te^{-t}$. In comparison, the fraction of isolated nodes is $c_1 \simeq e^{-t}$. Interestingly, the properly normalized correlation between the node degree and the component size is time independent, $\langle kl\rangle/\langle k\rangle\langle l\rangle = 2$. The node degree and the component size are correlated: nodes that have more links likely belong to larger components.

Paths

Since every two nodes in a component are connected, there must be a path connecting them. Let $p_{l,k}$ be the density of paths of length l in components of size k. There is the obvious bound $0 \le l \le k-1$ and additionally, there is a sum rule $\sum_l p_{l,k} = k^2 c_k$ reflecting that there are k^2 distinct paths in a component of size k. The density of linkless paths is $p_{0,k} = kc_k$.

The path length and the component size separately undergo an aggregation process and combining the two processes, these two indices undergo a bi-aggregation process.

The joint distribution evolves according to the rate equation

$$\frac{dp_{l,k}}{dt} = \sum_{\substack{i+j=k \\ n+m=l-1}} p_{n,i} p_{m,j} + \sum_{i+j=k} (ip_{l,i})(jc_j) - kp_{l,k}. \tag{23}$$

The initial conditions are $p_{l,k}(0) = \delta_{k,1}\delta_{l,0}$. There are two separate convolutions: one over the path length and one over the component size. The first term on the right-hand side of Eq. (23) describes newly formed paths due to linking and the last two terms correspond to paths that do not contain the newly placed link.

The path length density is

$$p_{l,k} = (l+1)\,\frac{k^{k-l-2}}{(k-l-1)!}\,t^{k-1}\,e^{-kt}. \tag{24}$$

The two shortest paths satisfy $p_{0,k} = kc_k$ and $p_{1,k} = 2(k-1)c_k$. The latter reflects that there are $k-1$ links in a (tree) component of size k. Also, the longest possible path, $l = k-1$, corresponds to a linear (chain-like) component, and the density of such components, $p_{k-1,k} = t^{k-1}e^{-kt}$, decays exponentially with length, so that such components are typically small.

The path length density can be simplified in the limit $k \gg l \gg 1$,

$$p_{l,k} \simeq l\,(2\pi k^3)^{-1/2}\,t^{k-1}\,e^{k(1-t)}\,e^{-l^2/2k}. \tag{25}$$

As was the case for the component size distribution, the path length density is self-similar in the vicinity of the percolation point, $p_{l,k} \to (1-t)^2\Phi_p\big(k(1-t)^2, l(1-t)\big)$, with the scaling function

$$\Phi_p(x,y) = y\,(2\pi x^3)^{-1/2}\exp(-y^2/2x). \tag{26}$$

The characteristic path length is as in (5) and the characteristic component size is as in (15). At the percolation point, the path length density (25) is governed by the factor $\exp(-l^2/2k)$ and therefore, the typical path length scales as square root of the component size

$$l \sim k^{1/2}. \tag{27}$$

The generating function $p(z,w) = \sum_{l,k} e^{kz} w^l p_{l,k}$ is expressed in terms of the auxiliary function (17)

$$p(z,w) = t^{-1}\frac{G(z+\ln t - t)}{1-wG(z+\ln t - t)}. \tag{28}$$

The total density of paths in finite components $p_{\rm tot} \equiv \sum_{l,k} p_{l,k}$ is therefore $p_{\rm tot} = u/t(1-u)$ and for $t<1$ we recover $p_{\rm tot} = 1/(1-t)$. Expanding $p(z,w)$ in powers of w, the total number of paths of length l, $p_l \equiv \sum_k p_{l,k}$, is given by $p_l = t^{-1}u^{l+1}$ with u satisfying (18), in accord with (4) for $t<1$.

Cycles

We have seen that the cycle length distribution is coupled to the path length distribution. In a similar way, the joint distribution of cycles in finite components of a given size is coupled to the joint distribution of paths of a given length in components of a given size.

To characterize cycles in a given component size, we consider the joint distribution $q_{l,k}$, the average number of components of size k containing a cycle of length l with $1 \leq l \leq k$. This joint distribution evolves according to the linear rate equation

$$\frac{dq_{l,k}}{dt} = \frac{1}{2} p_{l-1,k} + \sum_{i+j=k} (iq_{l,i})\,(jc_j) - k\,q_{l,k} \tag{29}$$

for $l \geq 1$. Initially, there are no cycles, and therefore $q_{l,k}(0) = 0$. The first term on the right hand side represents generation of cycles from paths, and the next two terms represent merger events where only the component size changes.

The joint cycle-length component-size distribution is

$$q_{l,k}(t) = \frac{1}{2} \frac{k^{k-l-1}}{(k-l)!} t^k e^{-kt}. \tag{30}$$

The smallest cycle, $l = 1$, is a self-connection, and the average number of such cycles is $q_{1,k} = \frac{t}{2} kc_k$. The largest cycles are rings, $l = k$, and their total number is on average $q_{k,k} = \frac{1}{2k} t^k e^{-kt}$. As for linear chains, the number of rings decays exponentially with length.

The large-k behavior of the cycle length distribution is similar to (25)

$$q_{l,k}(t) \simeq (8\pi k^3)^{-1/2}\, t^k\, e^{k(1-t)}\, e^{-l^2/2k}. \tag{31}$$

This distribution is self-similar in the vicinity of the percolation transition, $q_{l,k}(t) \to (1-t)^3 \Phi_q\left(k(1-t)^2, l(1-t)\right)$, with the scaling function $\Phi_q(x,y) = (8\pi x^3)^{-1/2} \exp(-y^2/2x)$. We see that the cycle length is characterized by the same scale as the path length, $l \sim (1-t)^{-1}$. At the percolation point, the cycle length distribution (31) is dominated by the factor $\exp(-l^2/2k)$ so that when the component size is fixed, the typical cycle length behaves as the typical path length, $l \sim k^{1/2}$. Moreover, the size distribution of finite components containing a cycle, $q_k = \sum_l q_{l,k}$, decays as a power-law at the percolation point, $q_k \simeq (4k)^{-1}$.

The joint generating function, $q(z,w) = \sum_{l,k} e^{kz}\, w^l\, q_{l,k}$, is

$$q(z,w) = \frac{1}{2} \ln \frac{1}{1 - wG(z + \ln t - t)}. \tag{32}$$

As for paths, statistics of cycles are directly coupled to statistics of components via the generating function $G(z)$. The total number of cycle-containing components of finite-size, $q_{\text{tot}} = \sum_{l,k} q_{l,k}$, is therefore $q_{\text{tot}}(t) = \frac{1}{2} \ln \frac{1}{1-u}$. Below the percolation point,

$q_{\rm tot}(t) = \frac{1}{2} \ln \frac{1}{1-t}$, for $t < 1$. Moreover, expanding $q(z,w)$ in powers of w shows that the the cycle length distribution (in finite components only) is $Q_l = u^l/2l$, in agreement with (7) prior to the percolation time ($t < 1$).

FINITE GRAPHS

Thus far, we used rate equations to describe infinite systems. While the rate equation approach can be extended to finite systems, the resulting equations are difficult to handle [25, 26]. When the number of nodes is finite, fluctuations are no longer negligible, and instead of a deterministic rate equation approach, a stochastic approach is needed. Finite-size scaling laws can be conveniently obtained by combining the exact infinite system results with scaling and extreme statistics arguments.

Finite-size scaling laws of random graphs are quite interesting. For example, the giant component nucleates at a size that it much smaller than the system size. The size of the largest component in the system, M, can be estimated by employing the cumulative component size distribution and the extreme statistics criterion, $N\sum_{k\geq M} c_k(t=1) \sim 1$. Using $c_k \sim k^{-5/2}$ gives

$$M \sim N^{2/3}. \tag{33}$$

The largest component in the system grows sub-linearly with the system size [3]. This component nucleates very close to the percolation time. The time τ when this component emerges approaches unity for large enough systems as implied by the diverging characteristic size scale $M \sim (1-\tau)^{-2}$, so that

$$1-\tau \sim N^{-1/3}. \tag{34}$$

Just past the percolation time, the size of the giant component grows linearly with time.

For finite systems, the scaling laws for the typical path length (5) combined with the characteristic component size (33) yields a scaling law for the characteristic path (and cycle) length

$$l \sim N^{1/3}. \tag{35}$$

One can deduce several other scaling laws and finite-size scaling functions underlying the path length density. For example, substituting the percolation time (34) into the total number of paths $P_{\rm tot} = (1-t)^{-1}$ yields the total path density $P_{\rm tot} \sim N^{1/3}$. Similarly, the total number of cycles at the percolation point grows logarithmically with the system size, $Q_{\rm tot}(N) \simeq \frac{1}{6} \ln N$.

In finite systems, it is possible that no cycle are created by the percolation time. This probability decreases algebraically with the system size, as seen from (9) and (34)

$$S \sim N^{-1/6}. \tag{36}$$

Moreover, combining the size distribution of the first cycle, $G_l(1) \sim l^{-3/2}$ with the characteristic cycle scale $l \sim N^{1/3}$ yields the moments of the size distribution corresponding to the first cycle

$$\langle l^n \rangle \sim N^{n/3-1/6}. \tag{37}$$

In particular, the average size of the first cycle is much smaller than the characteristic cycle length $\langle l \rangle \sim N^{1/6}$.

SUMMARY

In summary, we used kinetic theory to describe structural properties of random graphs including paths, cycles, and components. Modeling the linking process dynamically shows that paths and components undergo separate aggregation processes. Cycles are generated by paths and thus, the cycle length distribution is coupled to the path length distribution.

Generally, size distributions decay exponentially away from the percolation point, but at the percolation point, algebraic tails emerge. As the system approaches this critical point, the size distributions follow a self-similar behavior and they are characterized by diverging size scales.

The kinetic theory approach is well-suited for treating infinite systems. Nevertheless, the behavior of finite systems can be obtained from heuristic scaling and extreme statistics arguments. This yields scaling laws for the typical component size, path length, and cycle length at the percolation point.

The rate equation approach is powerful in that it utilizes a continuous time variable and therefore, differential, rather than difference equations. It has been successfully used to model growing random networks and it should be applicable to more complex random structures.

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