Random Geometric Series

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Integer sequences where each element is determined by a previous randomly chosen element are investigated analytically. In particular, the random geometric series $x_n = 2x_p$ with $0 \le p \le n-1$ is studied. At large n, the moments grow algebraically, $\langle x_n^s\rangle \sim n^{\beta(s)}$ with $\beta(s) = 2^s - 1$, while the typical behavior is $x_n \sim n^{\ln 2}$. The probability distribution is obtained explicitly in terms of the Stirling numbers of the first kind and it approaches a log-normal distribution asymptotically.

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I. INTRODUCTION

Integer sequences are ubiquitous in pure and applied mathematics, physics, and computer science [1]. While traditional integer sequences are deterministic, there is a growing interest in stochastic counterparts of fundamental sequences and their relevance to disordered or random systems. For example, the random Fibonacci sequence $x_n = x_{n-1} \pm x_{n-2}$ [2–4] has links with various topics in condensed matter physics, dynamical systems, products of random matrices, etc. (see e.g. [5–10]).

Random integer sequences are conceptually simple, yet they exhibit a complex phenomenology resulting from the memory generated by the stochastic recursion law. For the random Fibonacci sequence, the typical behavior is $x_n \sim e^{\lambda n}$ with the intriguing Lyapunov exponent $\lambda = 0.12397559$; furthermore, the distribution of the ratio x_n/x_{n-1} has singularities at every rational value [3, 4] and the model exhibits a remarkably intricate spectrum [10]. Also, the typical growth is different than the average growth as characterized by the moments $\langle x_n^s\rangle$.

Another form of randomness in which an element in the series depends on two previous elements, at least one of which is chosen randomly, was introduced recently [11]. For example, the stochastic Fibonacci-like series defined recursively by the rule $x_n = x_{n-1} + x_p$ with p randomly chosen between 0 and $n-1$, exhibits the typical growth $x_n \sim \exp(\lambda\sqrt{n})$ with the (numerically calculated) Lyapunov exponent $\lambda = 1.889$. The moments exhibit multiscaling [11, 12] and the probability distribution becomes log-normal asymptotically. Small variations in the recurrence rule can lead to substantial changes in the sequence characteristics — the growth law may be algebraic, log-normal, or exponential and the distribution may or may not exhibit multiscaling of the moments. In this article, we show that much of this phenomenology is captured by even simpler stochastic series, for which a more detailed analytical treatment is feasible.

II. RANDOM MULTIPLICATIVE SERIES

We consider recursively defined series where an element is determined by a *single* previously chosen element. A natural starting point is the random geometric series

$$x_n = 2x_p, \tag{1}$$

when $n \ge 1$ and $x_0 = 1$. The index p is chosen randomly between 0 and $n-1$ at each step. The first element is $x_0 = 1$. For example, for $n \le 2$, there are two, equally probable sequences: $x_n = 1, 2, 2$ or $1, 2, 4$. The series may not necessarily be monotonic.

A. The Moments

The moments $\langle x_n^s\rangle$ can be obtained analytically. They obey the recursion relation

$$\langle x_n^s\rangle = \frac{2^s}{n}\sum_{p=0}^{n-1}\langle x_p^s\rangle \tag{2}$$

for $n \ge 1$, with $\langle x_0^s\rangle = 1$. This recursion is solved using the generating function $M(s,z) = \sum_{n=0}^{\infty}\langle x_n^s\rangle\, z^n$. The recursion relation (2) leads to the ordinary differential equation $\frac{dM}{dz} = \frac{2^s}{1-z}M$ subject to the boundary condition $M(s,0) = 1$. Expanding the solution

$$M(s,z) = (1-z)^{-2^s} \tag{3}$$

in powers of z gives the moments

$$\langle x_n^s\rangle = \frac{\Gamma(n+2^s)}{\Gamma(2^s)\,\Gamma(n+1)}. \tag{4}$$

The first and the second moment are given by simple polynomials

$$\langle x_n\rangle = n+1, \quad \langle x_n^2\rangle = \frac{(n+1)(n+2)(n+3)}{6}. \tag{5}$$

Generally, there is a series of special values of s for which the moments are polynomial in n. For $2^s = k$ with k being an integer, the moments are

$$\langle x_n^{\ln k/\ln 2}\rangle = \frac{(n+1)(n+2)\cdots(n+k-1)}{(k-1)!}. \tag{6}$$

Asymptotically, all moments grow algebraically

$$\langle x_n^s\rangle \simeq A(s)\, n^{\beta(s)} \tag{7}$$

with $\beta(s) = 2^s - 1$ and $A(s) = 1/\Gamma(2^s)$. Therefore, the moments exhibit a multiscaling behavior characterized by the nonlinear spectrum of exponents $\beta(s)$.

B. The Probability Distribution

The probability distribution of the random variable x_n and its typical behavior are obtained by considering a closely related random sequence. Since the spectrum of possible values for x_n is 2^m with integer $m \geq 0$, we study the variable $m_n = \log_2 x_n$. The corresponding random additive series obeys the recursion relation

$$m_n = m_p + 1 \tag{8}$$

with $m_0 = 0$ and a randomly chosen $0 \leq p \leq n-1$. Generally, $1 \leq m_n \leq n$ for $n \geq 1$. The probability distribution $P_{n,m} = \mathrm{Prob}(x_n = 2^m)$ satisfies

$$P_{n,m} = \frac{1}{n}\sum_{l=0}^{n-1} P_{l,m-1} \tag{9}$$

for $n \geq 1$ and $P_{0,m} = \delta_{m,0}$. From this recursion, one readily obtains $nP_{n,m} - (n-1)P_{n-1,m} = P_{n-1,m-1}$, thereby recasting (9) into

$$P_{n,m} = \frac{n-1}{n}P_{n-1,m} + \frac{1}{n}P_{n-1,m-1}\,. \tag{10}$$

To tackle this recursion it is convenient to eliminate the denominator. The modified distribution $G_{n,m} = n!P_{n,m}$ satisfies the recursion

$$G_{n,m} = (n-1)G_{n-1,m} + G_{n-1,m-1} \tag{11}$$

with $G_{0,m} = \delta_{m,0}$. The very same recurrence generates $\left[{n \atop m}\right]$, the Stirling numbers of the first kind [13]. These numbers are closely related to the binomial coefficients and appear in numerous applications [14–16].

Thus $G_{n,m} = \left[{n \atop m}\right]$, and the probability distribution is expressed in terms of these special numbers as follows:

$$P_{n,m} = \frac{1}{n!}\begin{bmatrix} n \\ m \end{bmatrix}. \tag{12}$$

Moments of the variable m_n are obtained from the generating function [17] satisfied by the Stirling numbers of the first kind [13]

$$S_n(w) = \sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix} w^m = w(w+1)\ldots(w+n-1). \tag{13}$$

Taking the logarithmic derivative gives the average

$$\langle m_n\rangle = \frac{d}{dw}\,\ln S_n(w)|_{w=1} = H_n \tag{14}$$

in terms of the harmonic numbers $H_n = \sum_{j=1}^n \frac{1}{j}$. Using the large n asymptotics of the harmonic numbers [13], we conclude that the average grows logarithmically

$$\langle m_n\rangle = \ln n + \gamma + \frac{1}{2n} + \cdots\,. \tag{15}$$

The second derivative $\frac{d^2}{dw^2}\ln S_n(w)|_{w=1}$ similarly gives $\langle m_n(m_n-1)\rangle$. The variance, $w_n^2 = \langle m_n^2\rangle - \langle m_n\rangle^2$, follows

$$w_n^2 = H_n - H_n^{(2)}. \tag{16}$$

Here, $H_n^{(2)} = \sum_{j=1}^n \frac{1}{j^2}$ are the second-order harmonic numbers. Asymptotically, the variance grows logarithmically

$$w_n^2 = \ln n + \gamma - \frac{\pi^2}{6} + \frac{3}{2n} + \cdots\,. \tag{17}$$

The leading asymptotic behavior of the distribution can be evaluated as well. Using properties of the Stirling numbers, the distribution for small m reads

$$\begin{aligned} P_{n,1} &= \frac{1}{n}, \\ P_{n,2} &= \frac{1}{n}\,H_{n-1}, \\ P_{n,3} &= \frac{1}{2n}\,\left[H_{n-1}^2 - H_{n-1}^{(2)}\right]. \end{aligned} \tag{18}$$

These exact results reflect that the distribution is Poissonian for sufficiently small m:

$$P_{n,m} \simeq \frac{1}{n}\,\frac{(\ln n)^{m-1}}{(m-1)!}. \tag{19}$$

The Poissonian form corresponds to the small-m tail of the distribution. To obtain the distribution for typical, rather than extremal, values of m, we consider the continuum limit of the recursion relation (10) where the distribution satisfies

$$n\frac{\partial P}{\partial n} + \frac{\partial P}{\partial m} = \frac{1}{2}\,\frac{\partial^2 P}{\partial m^2}. \tag{20}$$

The change of variables $n \to t = \int_1^n dn'/n'$ transforms (20) into the standard diffusion-convection equation whose solution admits a Gaussian form

$$P_{n,m} \to \frac{1}{\sqrt{2\pi w_n^2}}\,\exp\left[-\frac{(m_n - \langle m_n\rangle)^2}{2w_n^2}\right]. \tag{21}$$

As $m_n = \ln x_n/\ln 2$, the distribution of x_n is therefore log-normal. Moreover, the variance of the random variable $\ln x_n$ is simply $(\ln 2)^2 \ln n$.

Both the Poissonian and the Gaussian behaviors follow from the more general asymptotic form of $P_{n,m}$

$$P_{n,m} \simeq \frac{1}{\Gamma(m/\ln n)}\,\frac{(\ln n)^m}{n\cdot m!} \tag{22}$$

that holds when $m \to \infty$ and $n \to \infty$ with the ratio $m/\ln n$ being finite. Indeed, using the asymptotic relation $\Gamma(x) \to x^{-1}$ as $x \to 0$ one recovers (19); the peak of the distribution (22) is at $m = \ln n$ and expansion in the vicinity of this peak recovers (21). We term the distribution (22) the modified Poissonian distribution.

To derive (22), we use (12)–(13) and the Cauchy theorem to express $P_{n,m}$ as an integral

$$P_{n,m} = \frac{1}{2\pi i}\oint \frac{dw}{w^{m+1}}\,\frac{w(w+1)\dots(w+n-1)}{n!} \tag{23}$$

over an arbitrary simple closed contour enclosing the origin in the complex w plane. When $n \to \infty$, the contour integral is easily computed by applying the steepest descent method. The saddle point is determined from

$$\frac{m}{w_*} = \sum_{j=1}^{n} \frac{1}{w_* + j}\,. \tag{24}$$

Asymptotically, $w_* \simeq m/\ln n$. We now deform the integration contour to the contour of steepest descent that runs through the saddle point along the imaginary axis (in the complex w plane). Writing $w = w_*(1+iy)$ and taking into account that the dominant contribution is gathered near $y = 0$, we obtain

$$\begin{aligned} P_{n,m} &\simeq \frac{w_*}{2\pi}\,\frac{(w_*+1)\dots(w_*+n-1)}{w_*^m\cdot n!}\int_{-\infty}^{\infty} dy\, e^{-my^2/2} \\ &= \frac{1}{\sqrt{2\pi m}}\,\frac{\Gamma(w_*+l)}{w_*^m\,\Gamma(n+1)\,\Gamma(w_*)} \\ &\simeq \frac{1}{\sqrt{2\pi m}}\,\frac{n^{w_*-1}}{w_*^m\,\Gamma(w_*)} \end{aligned}$$

where we used two properties of the gamma function — the difference equation $\Gamma(x+1) = x\,\Gamma(x)$ and the asymptotic relation $\frac{\Gamma(x+a)}{\Gamma(x)} \to x^a$ as $x \to \infty$. Inserting $w_* \simeq m/\ln n$ and using the Stirling formula leads to (22).

C. The Typical Behavior

The asymptotic distribution (21) and the growth law (15) lead to the typical behavior

$$x_n \simeq C\, n^{\ln 2} \tag{25}$$

with $C = 2^\gamma \cong 1.491967$ and $\gamma \cong 0.577215$ the Euler's constant. Since asymptotically, $\ln x_n \to \langle \ln x_n\rangle$, the typical behavior (25) emerges from the $s \to 0$ limit of the properly modified moments $\langle x_n^s\rangle^{1/s}$ in Eq. (7). In other words, the Lyapunov exponent $\lambda = \ln 2$, defined

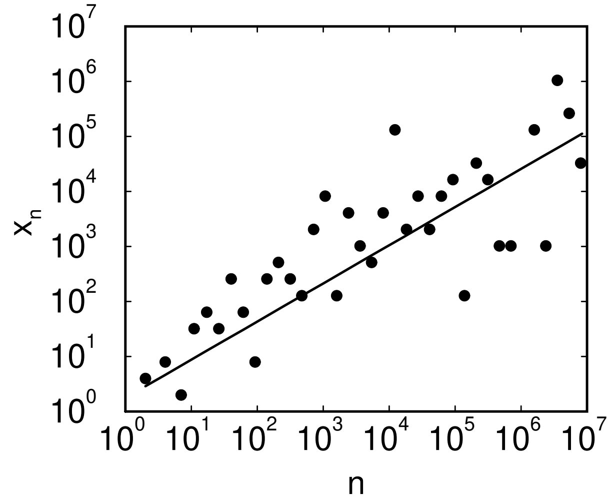

FIG. 1: A single realization of the random geometric series (bullets) versus the typical behavior (25), shown using a line. For clarity, only a small fraction of the series elements are displayed.

via $x_n \sim \exp(\lambda \ln n)$ is obtained from the moment spectrum using $\lambda = \lim_{s\to 0} s^{-1}\beta(s)$. However, the typical behavior (25) and the asymptotic distribution (21) do not yield the moments as they imply the quadratic moment spectrum $s\ln 2 + \frac{1}{2}(s\ln 2)^2$, equal to the first two terms in the Taylor expansion of $\beta(s)$.

There are large fluctuations between successive elements in a given series and large series-to-series variations. The typical behavior is eventually approached but very slowly, as illustrated in Fig. 1.

Let us compare random geometric series with random Fibonacci-like series. Overall, the behavior is in line with the behavior found for the series $x_n = x_{n-1} + x_p$ with $0 \le p \le n-1$. In both cases, the distribution of x_n is log-normal and the moments exhibit multiscaling [11]. In the present case, it is possible to find the Lyapunov exponent. However, the behavior is unlike the one found for the random sequence $x_n = x_p + x_q$ with $0 \le p, q \le n-1$ despite the fact that in both cases the average is $\langle x_n\rangle = n+1$. The average characterizes all the moments and the distribution approaches an ordinary scaling form $P_n(x) \to n^{-1}\Phi(xn^{-1})$ [11].

D. Extremal Statistics

The span of the additive random sequence, i.e. the set of all possible values of m, provides an additional statistical characterization. In every realization, this set contains no gaps, so the span is equivalent to the largest sequence element M_n. Thus, the span is directly related to extremal characteristics of the sequence. To find how M_n grow with n, it is necessary to consider the large m tail of the probability distribution outside the Gaussian region. The modified Poissonian distribution (22) suggests that $M_n \sim \ln n$.

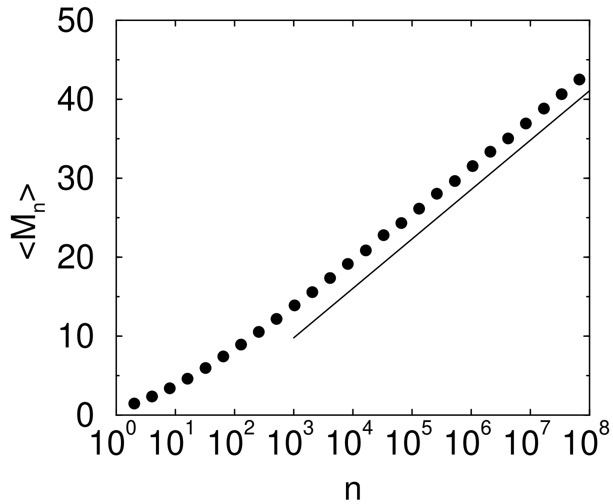

FIG. 2: The maximal element in the series. The average $\langle M_n \rangle$, obtained from 100 independent realizations of the random additive series (bullets) is compared with the heuristic estimate (28), shown using a line of slope e.

We obtain the growth of the maximal element in the series heuristically using the extreme value criterion

$$\sum_{n'=M_n}^{n} \sum_{m'=M_n}^{n'} P_{n',m'} \sim 1. \tag{26}$$

Since the distribution $P_{n,m}$ quickly diminishes with m in the tail region, this extreme statistics criterion becomes $\sum_{n'=M_n}^{n} P_{n',M_n} \sim 1$, and as $M_n \ll n$, one has

$$nP_{n,M_n} \sim 1. \tag{27}$$

Combining this criterion with the distribution (22) and using the Stirling formula, we arrive at the following asymptotic growth of the maximal value

$$M_n \simeq e \ln n. \tag{28}$$

Numerical simulations are in good agreement with this estimate (Fig. 2). A more rigorous derivation including the leading correction is given in Appendix A.

This growth is ultimately connected with the frequency by which the largest element in the sequence occurs. At the nth step, the maximal value M_n is augmented by one with probability h/n with h the frequency by which the largest element occurs. Thus, $\langle M_n \rangle = \langle M_{n-1} \rangle + \frac{\langle h \rangle}{n}$. This leads to the growth law $\langle M_n \rangle \simeq \langle h \rangle \ln n$. The frequency h is a random variable that in principle depends on n yet it has a stationary distribution $p(h)$ in the large n limit. The growth law (28) implies $\langle h \rangle = \sum hp(h) = e$.

III. GENERALIZATIONS

There are a number of natural generalizations of the stochastic sequences (1) and (8). Below, we briefly describe two examples. In both cases we consider the variable m_n directly.

A. Random Random Walk

In the random geometric series, the previous element is chosen randomly while the recursive rule is deterministic. We thus consider the *stochastic* recursion relation

$$m_n = m_p \pm 1 \tag{29}$$

with $m_0 = 0$ and a randomly chosen $0 \le p \le n-1$ on the $n^{\rm th}$ step. We assume that both signs in (29) are taken with equal probability. We term the sequence generated by (29) the random random walk.

The moments can be obtained recursively, as in the random geometric series and we merely quote the results. The first moment vanishes, $\langle m_n \rangle = 0$, and the variance is given by the harmonic numbers

$$w_n^2 = H_n. \tag{30}$$

The asymptotic behavior is therefore $w_n^2 \simeq \ln n$.

The probability $P_{n,m}$ that the walk is at position m at the $n^{\rm th}$ step obeys

$$P_{n,m} = \frac{n-1}{2n} P_{n-1,m} + \frac{1}{2n}\left[P_{n-1,m-1} + P_{n-1,m-1}\right]. \tag{31}$$

Taking the continuum limit, we find that the distribution satisfies $n\,\frac{\partial P}{\partial n} = \frac{\partial^2 P}{\partial m^2}$. This diffusion equation shows that the distribution is Gaussian

$$P_{n,m} \simeq \frac{1}{\sqrt{2\pi w_n^2}} \exp\left[-\frac{m^2}{2w_n^2}\right]. \tag{32}$$

Therefore, all moments are characterized by the variance: $\langle m_n^{2k} \rangle \simeq (2k-1)!!(\ln n)^k$. The random random walk spreads very slowly with the typical spread

$$m \sim \sqrt{\ln n}. \tag{33}$$

Hence, the first passage time, the time to reach a site of distance m from the origin grows as $\exp(m^2)$. What remains an open question is whether the distribution $P_{n,m}$ can be obtained in a closed form from the generating function $\sum_{n,m} z^n w^m P_{n,m} = (1-z)^{-w(w+1)}$. In terms of the variable $x_n = 2^{m_n}$, the growth is slower than any power law, $x_n \sim \exp(\ln 2\sqrt{\ln n})$.

B. Two Dimensions

Thus far, we considered only one-dimensional sequences. In physical systems, it is generally believed that disorder, no matter how small, is always relevant in two-dimensions [6, 7]. However, disordered two-dimensional systems are typically untreatable analytically.

We consider a natural generalization of (8) to a two-dimensional square lattice. Starting from $m_{\mathbf{0}} = 1$, where $\mathbf{0} = (0,0)$ is the site at the origin, the values at further sites are determined recursively according to

$$m_{\mathbf{n}} = m_{\mathbf{p}} + 1 \tag{34}$$

Here $\mathbf{p} = (p_1, p_2)$ is chosen equiprobably among lattice sites that are closer to the origin than $\mathbf{n}$, i.e. $|\mathbf{p}| < |\mathbf{n}|$;. We choose the "manhattan distance" from the origin $|\mathbf{n}| = |n_1| + |n_2|$ as the measure of distance.

The probability distribution depends only on the norm $n = |\mathbf{n}|$, so we keep the notation $P_{n,m}$. For the norm $n = |\mathbf{n}| = |n_1|+|n_2|$, there are $4n$ lattice sites a distance n from the origin, $1+4+\ldots+4(n-1) = 1+2n(n-1)$ lattice sites which are a distance $\leq n-1$ from the origin. The probability distribution satisfies the recursion relation

$$\begin{aligned} P_{n,m} &= \left[1 - \frac{4(n-1)}{1+2n(n-1)}\right] P_{n-1,m} \\ &+ \frac{4(n-1)}{1+2n(n-1)}\, P_{n-1,m-1} \end{aligned} \tag{35}$$

for $n \geq 2$ with $P_{0,m} = \delta_{m,0}$ and $P_{1,m} = \delta_{m,1}$.

In analogy with the one-dimensional case, we write the distribution in the form

$$P_{n,m} = \frac{1}{\Pi_n}\, G^{(2)}_{n,m} \tag{36}$$

with $\Pi_n = \prod_{j=1}^{n}[1+2j(j-1)]$ and $G^{(2)}_{n,m}$ the two-dimensional analogs of the Stirling numbers of the first kind. These non-negative integer numbers obey the fundamental recursion relation

$$G^{(2)}_{n+1,m} = [1+2n(n-1)]G^{(2)}_{n,m} + 4nG^{(2)}_{n,m-1} \tag{37}$$

for $n \geq 2$ and $G^{(2)}_{0,m} = \delta_{m,0}$, $G^{(2)}_{1,m} = \delta_{m,1}$. The corresponding generating function is

$$\sum_{m=0}^{n} G^{(2)}_{n,m}\, w^m = w \prod_{j=1}^{n} [1 + 2(j-1)(j-2+2w)] \tag{38}$$

for $n \geq 1$. Using this generating function, the average and the variance are

$$\begin{aligned} \langle m_n \rangle &= 1 + \sum_{j=1}^{n} \frac{4(j-1)}{1+2j(j-1)}\,, \\ w_n^2 &= \sum_{j=1}^{n} \frac{4(j-1)}{1+2j(j-1)} - \sum_{j=1}^{n} \left[\frac{4(j-1)}{1+2j(j-1)}\right]^2. \end{aligned} \tag{39}$$

The leading asymptotic behaviors are $\langle m_n \rangle \simeq 2\ln n$ and $w_n^2 \simeq 2\ln n$. In the continuum limit, the distribution obeys the diffusion-convection equation that now has the form $n\,\frac{\partial P}{\partial n} + 2\frac{\partial P}{\partial m} = \frac{\partial^2 P}{\partial m^2}$. Asymptotically, the distribution is Gaussian as in the one-dimensional case (21) with the average and the variance merely modified by the factor 2. One can also show, by generalizing the moment recursion relation (2), that the spectrum is also modified by the same factor: $\beta(s) = 2(2^s - 1)$.

IV. SUMMARY

In summary, we considered random sequences where an element depends on a previous randomly chosen element and have shown that they exhibit a similar phenomenology as sequences that involve dependence on a few previous elements. The typical behavior and the moment behavior provide a statistical characterization of the sequence. The growth laws depend sensitively on details of the recurrence relations.

We obtained a number of exact and asymptotically exact results for the probability distribution and its moments. For the random geometric series, the sequence growth is algebraic. The moments exhibit multi-scaling asymptotic behavior and also contain information regarding the typical behavior. Asymptotically, the probability distribution becomes log-normal but it does not fully characterize the actual moment behavior.

There are additional interesting questions that can be asked for this family of random series including the likelihood of monotonically increasing sequences, growth of correlations between two different elements in the same sequence, and statistics of the number of distinct elements in a given sequence.

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APPENDIX A: DERIVATION OF EQ. (28)

Substituting (22) into the criterion $\sum_{n'=M_n}^{n} P_{n',M_n} \sim 1$ and replacing the summation by integration yields

$$\begin{aligned} \sum_{n'=M_n}^{n} P_{n',M_n} &\simeq \frac{1}{(M_n)!} \int^{n} \frac{dn'}{n'}\, \frac{(\ln n')^{M_n}}{\Gamma(M_n/\ln n')} \\ &\sim \frac{(\ln n)^{M_n+1}}{(M_n+1)!} \sim 1. \end{aligned} \tag{A1}$$

Taking the logarithm of (A1) and using the Stirling formula, we obtain an implicit relation for the maximal element

$$M_n - M_n \ln\left(\frac{M_n}{\ln n}\right) = \frac{3}{2}\,\ln M_n - \ln\ln n. \tag{A2}$$

This yields the leading correction to Eq. (28)

$$M_n \to e\ln n - \frac{1}{2}\,\ln\ln n + \ldots. \tag{A3}$$

M_n is of course a random variable and (A3) represents its average $\langle M_n \rangle$. We anticipate that fluctuations in M_n remain finite as $n \to \infty$.

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