Dynamics of Social Diversity

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We introduce and solve analytically a model for the development of disparate social classes in a competitive population. Individuals advance their fitness by competing against those in lower classes, and in parallel, individuals decline due to inactivity. We find a phase transition from a homogeneous, single-class society to a hierarchical, multi-class society. In the former case, the population is uniformly poor. In the latter case, a finite-fraction condensate that consists of a static lower class remains. The rest of the population consists of an upwardly-mobile middle class, on top of which lies a tiny upper class in the form of a thin boundary layer.

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There are many connections between social dynamics and physical processes. For example, in urban dynamics [1], migration-driven population development has analogies with coarsening [2]. Rumor propagation and formation of social networks are closely related to percolation [3]. An appealing route for modeling social phenomena is to identify individuals in a society as particles in a physical system, *i.e.*, an agent-based description [4, 5]. This interdisciplinary approach has helped identify underlying mechanisms for fundamental social phenomena and has led to quantitative predictions [6–10].

In this spirit, we seek to understand the formation of the social hierarchies that are ubiquitously observed in animal populations [11, 12] and in human societies [13]. We introduce a minimalist agent-based model in which competition is the underlying mechanism for social differentiation. Using concepts and methods from statistical and nonlinear physics, such as scaling and asymptotic analysis, we find a rich phenomenology for social diversity. As a function of the competition rate, the population undergoes a phase transition from a homogeneous, single-class society to a hierarchical, multi-class society. In the latter phase, the lower class remains destitute and static and has the character of a condensate, while the middle class is dynamic and has a continuous upward mobility.

Our work is based on an earlier model of Bonabeau [14], in which each individual is endowed with a fitness-like variable that evolves by two opposing processes. The first is competition: when two agents interact, one individual becomes more fit (gains status) and the other becomes less fit, with the initially fitter individual being more likely to win. Counterbalancing this competition, the winning probability for the fitter agent decreases as the time from the last competition increases. This model was found to exhibit a transition to a heterogeneous society as the relative influence of competition is increased [14, 15].

In our model, we account for the interplay between advancement by competition and decline by inactivity via a single parameter. Each agent is endowed with an integer fitness value $k \geq 0$ that can change due to two processes: (i) advancement by competition and (ii) decline by inactivity. In the competition step, when two agents interact, their fitnesses change according to

$$(k, j) \to (k+1, j), \tag{1}$$

for $k \geq j$. When two equally fit agents compete, both advance [16]. Without loss of generality, the rate of this process is set to one. We also consider the mean-field limit where any pair of agents is equally likely to interact. The rationale behind this "rich gets richer" dynamics is obvious: fitter individuals are better suited for, and hence benefit from, competition. When decline occurs, individual fitness decreases as

$$k \to k-1 \tag{2}$$

with a rate r. This process reflects the natural tendency for social status to decrease in the absence of interactions. The lower limit for the fitness is $k = 0$; once an individual reaches zero fitness, there is no further decline. The model is characterized by a single parameter, the rate of decline r.

Let $f_k(t)$ be the fraction of agents with fitness k at time t. This distribution obeys the nonlinear master equation

$$\frac{df_k}{dt} = r(f_{k+1} - f_k) + f_{k-1}F_{k-1} - f_k F_k \tag{3}$$

for $k > 0$, and $df_0/dt = rf_1 - f_0^2$ for $k = 0$. The quantity $F_k = \sum_{j=0}^{k} f_j$ is the cumulative distribution. We take the initial condition to be $f_k(0) = \delta_{k,0}$. In Eq. (3), the first two terms account for decline, while the last two terms account for advancement [17].

To understand the behavior of this system, we focus on the cumulative distribution F_k, from which the individual densities are $f_k = F_k - F_{k-1}$. From the master equation (3), the cumulative distribution satisfies

$$\frac{dF_k}{dt} = r(F_{k+1} - F_k) + F_k(F_{k-1} - F_k), \tag{4}$$

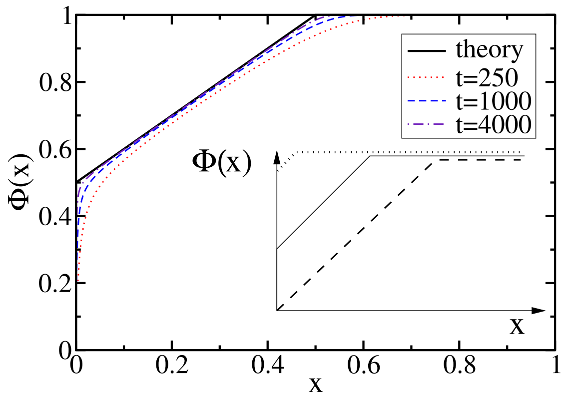

FIG. 1: The middle class. The scaled cumulative distribution $\Phi(x)$ versus x for $r = 1/2$ at $t = 250$ (dotted), 1000 (dashed), 4000 (dot-dashed). The solid line is the theoretical prediction (8). The inset shows the qualitative behavior for $r = 0$ (dashed), $r \approx 1/2$ (solid), and $r \approx 1$ (dotted).

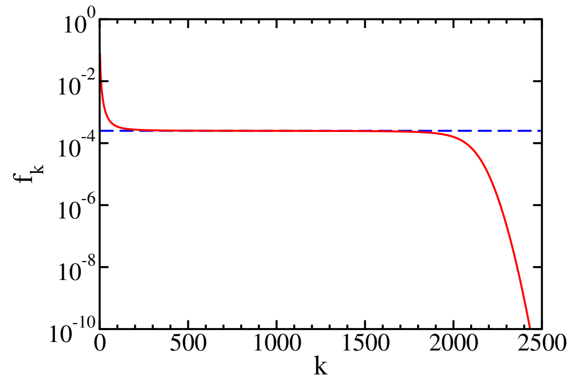

FIG. 2: The lower, middle, and upper classes. The fitness distribution f_k versus k for $r = 1/2$ at $t = 4000$ (solid line), showing the lower ($k \lesssim 60$), middle, and upper ($k \gtrsim 2000$) classes. Also shown for reference is the plateau $f_k = 1/t$ (dashed line). The distribution varies as $1/k^2$ in the lower class up to the diffusive scale $k_{\rm lower} \sim (2rt)^{1/2}$. The distribution is constant in the middle class up to a ballistic scale $k_{\rm upper} = (1-r)t$, beyond which there is an upper class that has a Gaussian decay.

for $k \geq 0$. The boundary condition is $F_{-1} = 0$ so that $dF_0/dt = r(F_1 - F_0) - F_0^2$, and the initial condition is $F_k(0) = 1$.

Homogeneous vs. Hierarchical Societies. Our social diversity model undergoes a phase transition from a homogeneous to a hierarchical society. This transition follows from the continuum limit of the master equation (4) for the cumulative distribution

$$\frac{\partial F}{\partial t} = (r - F)\frac{\partial F}{\partial k}. \tag{5}$$

For finite fitness, the cumulative distribution approaches a steady state in the long-time limit. Then either $F = r$ or $\partial F/\partial k = 0$. Invoking the bound $F \leq 1$, we conclude that either $F = r$ or $F = 1$. Therefore, L, the fraction of the population with finite fitness exhibits a phase transition

$$L = \begin{cases} r & r < 1; \\ 1 & r \geq 1. \end{cases} \tag{6}$$

When competition is weak, the entire population has a finite fitness, while for strong competition, only a fraction $L < 1$ of the population has a finite fitness.

We shall see that the quantity L is the size of the lower class, while the complementary fraction $1 - L$ is the size of the middle class, whose fitness increases indefinitely. Thus for $r \geq 1$, the society is homogeneous and consists of a single lower class. However for $r < 1$, there is a hierarchical society that contains a distinct lower class, and a distinct a middle class. When $r = 0$, the lower class disappears entirely.

Middle Class Dynamics. The picture presented above is confirmed by analyzing the dynamics of the middle class. Applying dimensional analysis to the governing Eq. (4) suggests that the characteristic fitness of the middle class increases linearly with time, $k \sim t$. Thus, we posit the scaling form

$$F_k \simeq \Phi(k/t) \tag{7}$$

with the boundary condition $\Phi(\infty) = 1$. Substituting Eq. (7) into (5), the scaling function satisfies $x\, d\Phi/dx = (\Phi - r)\, d\Phi/dx$ where $x = k/t$. The solution is either $\Phi(x) = r + x$ or $d\Phi/dx = 0$. As a result (Fig. 1)

$$\Phi(x) = \begin{cases} r + x & x < 1 - r; \\ 1 & x \geq 1 - r. \end{cases} \tag{8}$$

Remarkably, the scaling function for the cumulative distribution is piecewise linear and thus non-analytic. The analysis above implicitly assumes continuity of the scaling function and indeed, the cumulative distribution is expected to be continuous.

The scaling function (8) has a number of basic implications. First, the quantity $\Phi(0) = r$ is the fraction of the population that belongs to the lower class, confirming the prediction of Eq. (6). This behavior is reminiscent of a physical condensate, where a finite fraction of the population occupies the zero fitness (in scaled units) ground state. In this sense, the entire lower class is destitute. When competition only occurs ($r = 0$), the society consists of a continuously-improving middle class.

We can alternatively write the fitness distribution in the scaling form $f_k \simeq t^{-1}\phi(k/t)$. The corresponding scaling function is $\phi(x) = d\Phi/dx = r\delta(x) + 1$ for $x \leq 1 - r$ and $\phi(x) = 0$ otherwise. The middle class thus has a *constant* fitness distribution

$$f_k \simeq t^{-1}, \tag{9}$$

for $k < k_{\rm upper} = (1-r)t$. The lot of the middle class is constantly improving, as the fitness extends over a grow-

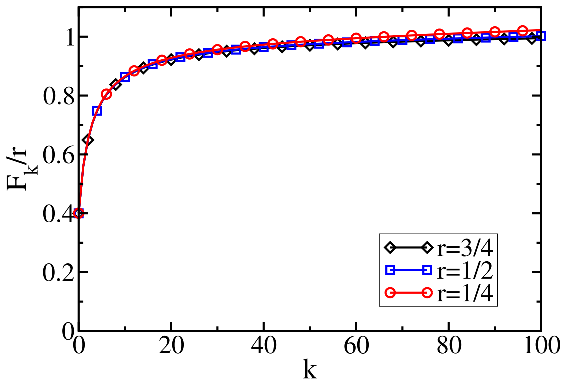

FIG. 3: The lower class. The cumulative distribution, normalized by the rate r, F_k/r is plotted versus k. Shown are simulation results for $r = 1/4$ (circles), $r = 1/2$ (squares), $r = 3/4$ (diamonds) at time $t = 10^4$.

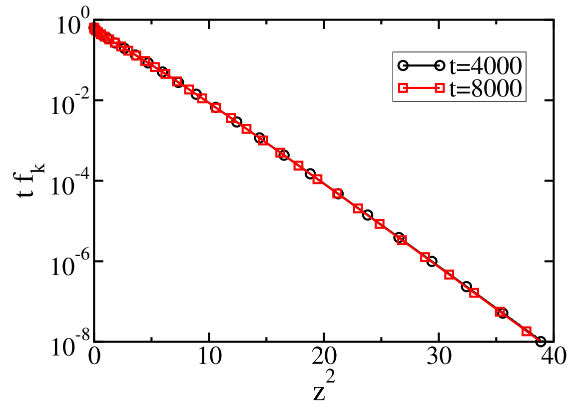

FIG. 4: The upper class. Shown is the normalized tail of the fitness distribution: tf_k versus z^2, with the scaling variable $z = (k - vt)/\sqrt{Dt}$, for $r = 1/2$ at times $t = 4000$ (circles) and $t = 8000$ (squares).

ing range and the average fitness increases linearly with time.

Numerical integration of the master equation confirms these predictions (Figs. 1 and 2). We used a fourth-order Adams-Bashforth method [18] with accuracy to 10^{-10} in the distribution F_k. Our numerical data was obtained by integrating F_k for $0 \le k < 20000$.

Lower Class Dynamics. The fitness of the lower class is finite; in other words, the fitness distribution is in a steady state. This distribution can be determined by setting the time derivative in the rate equation to zero. Writing $F_k = L(1 - G_k)$, so that the deviation G_k vanishes at large k, Eq. (4) gives

$$r\,\frac{G_{k+1} - G_k}{G_k - G_{k-1}} = L(1 - G_k). \tag{10}$$

The fitness distribution is fundamentally different in the two phases. In the homogeneous society phase ($r \ge 1$ and $L = 1$), the deviation G_k decays rapidly at large fitness. Replacing the right-hand side of Eq. (10) by 1 for large k, the solution is simply $G_k \sim r^{-k}$. Therefore

$$f_k \sim r^{-k}. \tag{11}$$

The fitness distribution decays exponentially, so that the lower class is confined to a small range of fitness values. The characteristic fitness $1/\ln r$ diverges as the transition is approached. The society is homogeneous with a single class, the lower class, that does not evolve with time.

In the hierarchical society phase, (where $r < 1$ and $L = r$), the fitness distribution is universal, as the recursion relation (10) becomes r-independent, $(G_{k+1} - G_k)/(G_k - G_{k-1}) = 1 - G_k$. This shows that F_k/r is a universal, r-independent distribution (Fig. 3). We start by treating k as a continuous variable, because the fitness range becomes large as $r \downarrow 1$. We thus expand the differences in Eq. (10) to second order. Since $G'' \ll G'$, where prime denotes differentiation with respect to k, we find $G'' + GG' = 0$. Integrating once and invoking $G \to 0$ as $k \to \infty$, gives $G' + \frac{1}{2}G^2 = 0$. Asymptotically, $G \simeq 2k^{-1}$, and using $f_k = F_k - F_{k-1}$, we find

$$f_k \simeq 2r\,k^{-2}. \tag{12}$$

The lower class has a power-law fitness distribution with mean fitness that diverges logarithmically in the upper limit. While the lower class is still static, it is not as destitute as in the homogeneous society phase.

The transition between the lower and middle class occurs when $2r/k^2 \approx 1/t$, *i.e.*, where the power-law distribution (12) matches the uniform distribution (9). Consequently, the lower class is confined to a diffusive boundary layer of thickness

$$k_{\rm lower} \sim (2rt)^{1/2}. \tag{13}$$

Beyond this diffusive scale, lies the middle class whose constant density (9) extends over the range $k_{\rm lower} < k < k_{\rm upper}$. In the hierarchical society phase, the fitness distribution consists of the stationary component (12) that defines the lower class and the evolving component (7) that defines the middle class. The extent of the stationary region indefinitely grows with time.

We thus conclude that the lower class is always static, being in a steady-state independent of the rate of decline r. In a homogeneous society, the lower class has an exponentially decaying fitness distribution that lies within a narrow fitness range. In a hierarchical society, the lower class fitness distribution decays algebraically and its range grows diffusively with time.

Upper Class Dynamics. The upper class is defined by the subpopulation whose fitness lies beyond $k_{\rm upper} = (1 - r)t$. We probe the tail of this fitness distribution by again considering the deviation G_k, defined by $F_k = 1 - G_k$. It obeys the Fokker-Planck equation

$$\frac{\partial G_k}{\partial t} + v\frac{\partial G_k}{\partial k} = D\frac{\partial^2 G_k}{\partial k^2} \tag{14}$$

with upward drift velocity $v = (1 - r)$ and diffusion coefficient $D = (1 + r)/2$. The boundary condition $G(k = vt) \propto t^{-1}$ is set by matching the density at the top of the middle class with that at the bottom of the upper class. Consequently, the fitness distribution, $f = -\partial G/\partial k$, follows the scaling form (Fig. 4)

$$f_k(t) \simeq t^{-1}\psi\left(\frac{k - vt}{\sqrt{Dt}}\right). \tag{15}$$

The scaling function has the Gaussian tail $\psi(z) \sim \exp(-z^2/2)$, as $z \to \infty$, characteristic of a convection-diffusion equation. The upper class is thus confined to a diffusive boundary layer that grows as $\sqrt{Dt}$. From Eq. (15), the upper class contains a fraction $\propto 1/\sqrt{t}$ of the total population.

For completeness, we note that for the special case of $r = 0$, the rate equation $dF_k/dt = F_k(F_{k-1} - F_k)$ admits an exact solution. We make the transformation

$$F_k = \frac{P_{k-1}}{P_k}, \tag{16}$$

with the initial condition $P_k(0) = 1$ and the boundary condition $P_{-1} = P_0 = 1$. Remarkably, this transformation reduces the non-linear rate equations to the set of linear equations $dP_k/dt = P_{k-1}$ for $k \geq 1$. Solving these recursively, we obtain $P_k = \sum_{j=0}^{k} t^j/j!$. Therefore,

$$F_k(t) = \frac{1 + t + \frac{1}{2!}t^2 \cdots + \frac{1}{k!}t^k}{1 + t + \frac{1}{2!}t^2 \cdots + \frac{1}{(k+1)!}t^{k+1}}. \tag{17}$$

It is possible to show that this exact solution adheres to the scaling form (7) with $\Phi(x)$ as in (8). Asymptotic analysis yields the exact shape of F_k in the boundary layer, $1 - F_k \simeq \sqrt{2/\pi t}\exp[-(k-t)^2/2t]/\mathrm{erfc}[(k-t)/\sqrt{2t}]$.

In summary, we introduced a minimal model of social diversity in which the two driving mechanisms are advancement by competition and decline by inactivity. An idealized but plausible social structure emerges: either a homogeneous society with a single lower class, or a hierarchical society with multiple classes. The lower class is always static, while the middle class and the tiny upper classes are upwardly mobile. In a hierarchical society, the lower and the upper classes are confined to boundary layers that are much smaller than the dominant scale that characterizes the fitness of the middle class. It is striking that a deceptively simple master equation exhibits such a rich structure, with a stationary component, followed by two transient components, as well as a non-analytic scaling function for the asymptotic fitness distribution.

There are numerous interesting questions suggested by this work. For example, what is the time history of an individual? How rigid is the social hierarchy and how does it depend on the population size? What happens if each individual is also endowed with an intrinsic fitness? Last, does non-trivial spatial organization emerge when agents move locally in space?

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