Stable Distributions in Stochastic Fragmentation

P. L. Krapivsky[1], E. Ben-Naim[2], and I. Grosse[3]
[1] *Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215 USA*
[2] *Theoretical Division and Center for Nonlinear Studies,*
Los Alamos National Laboratory, Los Alamos, NM 87545 USA
[3] *Cold Spring Harbor Laboratory, Cold Spring Harbor, NY 11724 USA*

We investigate a class of stochastic fragmentation processes involving stable and unstable fragments. We solve analytically for the fragment length density and find that a generic algebraic divergence characterizes its small-size tail. Furthermore, the entire range of acceptable values of decay exponent consistent with length conservation can be realized. We show that the stochastic fragmentation process is non-self-averaging as moments exhibit significant sample-to-sample fluctuations. Additionally, we find that the distributions of the moments and of extremal characteristics possess an infinite set of progressively weaker singularities.

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I. INTRODUCTION

Fragmentation is a fundamental stochastic process with a variety of applications ranging from geology [1] and fracture [2] to the breakup of liquid droplets [3] and atomic nuclei [4, 5]. Fragmentation processes are also relevant to spin glasses, Boolean networks and genetic populations [6–11].

This study is motivated by *conditional* fragmentation processes where fragmentation may not proceed indefinitely. Instead, whether a fragment undergoes further breakage is a function of its characteristics. Examples include segmentation of sequences in algorithms such as genome reconstruction [12–14], DNA segmentation [15–17], and search algorithms [18] on the one hand and collision-induced fragmentation of solid objects [19–24] on the other hand. These applications lead to stochastic fragmentation processes where not only the way by which fragments are produced, but also the number of fragmentation events is subject to a random process.

In the simplest stochastic fragmentation process, one starts with a fragment and breaks it into two pieces. With probability p, a newly formed fragment remains unstable, i.e, it continues to participate in fragmentation events, while with probability $q = 1 - p$, it becomes stable, and is never fragmented again. The process is repeated for all unstable fragments until all fragments become stable. For this simple process, we show that the final length density, $P(x)$, is purely algebraic, namely $P(x) = 2qx^{-2p}$ [25]. Similar scale free behavior was observed in other fragmentation processes [23, 26]. Stochastic fragmentation also exhibits intriguing statistical characteristics including moments which are non-self-averaging, essential singularities in the distribution of the moments, and an infinite set of singularities in the distribution of the largest and the smallest fragments.

In this paper, we study statistical properties of stochastic fragmentation processes. In Sec. II, we introduce the stochastic fragmentation model and derive the fragment length density $P(x)$. We also consider generalizations to size dependent fragmentation densities and fragmentation probabilities, and obtain the exponent underlying the algebraic behavior of $P(x)$ as a root of a transcendental equation. In Sec. III, we show that the stochastic fragmentation process is non-self-averaging. Specifically, the moments $Y_\alpha = \sum_i x_i^\alpha$ exhibit significant sample-to-sample fluctuations. In Sec. IV, we study extremal characteristics such as the distribution of the largest and the shortest fragment, and show that both the extremal distributions and the distribution of the moments are characterized by an infinite set of progressively weaker singularities. In Sec. V, we present an application to random sequential adsorption processes, and we summarize this work in Sec. VI.

II. THE MODEL

In the basic stochastic fragmentation process, we start with the unit interval, which is considered to be unstable. This interval is fragmented into two pieces of length l and $1-l$, where l is drawn from a uniform probability density $\rho(l) = 1$. The newly formed fragment remains unstable with probability p, and with probability $q \equiv 1 - p$ it becomes stable and does not undergo further fragmentation. The process is iterated for unstable fragments until all fragments become stable.

The average total number of stable fragments, $\langle N\rangle$, can be directly evaluated. Consider a fragment produced in the first fragmentation event. With probability q it is stable, and consequently, only a single fragment is produced; otherwise, the process is repeated. Hence $\langle N\rangle = 2(q + p\langle N\rangle)$, yielding

$$\langle N\rangle = \begin{cases} 2q/(1-2p), & \text{if } p < 1/2; \\ \infty, & \text{if } p \geq 1/2. \end{cases} \tag{1}$$

The average total number of fragments diverges as the probability p approaches the critical point $p_c = 1/2$, reflecting the critical nature of the corresponding branching process [27].

Next, consider the fragment length density, $P(x)$, of stable fragments of length x. The recursive nature of the process can be used to obtain the governing equation for the fragment length density

$$P(x) = 2\left[q + p\int_x^1 \frac{dy}{y}\, P\left(\frac{x}{y}\right)\right]. \tag{2}$$

The first term accounts for stable fragments produced at the first generation, and the second term describes creation of an x-fragment from a larger y-fragment.

Equation (2) can be solved by employing the Mellin transform technique. Let

$$M(s) \equiv \int dx\, x^{s-1} P(x). \tag{3}$$

Hereinafter, integration with unspecified limits is carried over the unit interval. Equations (2) and (3) yield $M(s) = 2s^{-1}\,[q + pM(s)]$, and consequently, the Mellin transform of the length density reads

$$M(s) = \frac{2q}{s-2p}. \tag{4}$$

Note that the total length is conserved, $M(2) = 1$, and that the total number of stable fragments, $M(1) = \langle N\rangle$, is consistent with Eq. (1).

The length density can be obtained by inverting the Mellin transform

$$P(x) = 2q\, x^{-2p}. \tag{5}$$

Remarkably, the length density is purely algebraic over the entire range $0 < x < 1$. Generally, given an algebraic divergence near the origin, $P(x) \sim x^{-\gamma}$, length conservation provides the upper bound $\gamma < 2$, and as $0 < \gamma = 2p < 2$, the entire range of possible divergences is realized by tuning p.

Interestingly, scale free distributions were also found for a dual stochastic aggregation process where aggregates may turn stable after each aggregation event. In that case, the large size tail of the distribution decays algebraically [28]. We also note that algebraic distributions have been observed in a number of impact fragmentation experiments involving rods, spheres, bricks, etc., with the corresponding decay exponents typically ranging between 1 and 2 [19–21].

The fragment length density is not altered as the critical point $p_c = \frac{1}{2}$ is passed. Nevertheless, this point is characterized by a unique property. Starting from an interval of length L_0, we arrive at the generalized form of Eq. (5):

$$P(x) = 2qL_0^{2p-1}x^{-2p}. \tag{6}$$

Hence, at the critical point, $P(x)$ becomes independent of the initial interval length L_0.

In the above basic model both the fragmentation density and the fragmentation probability were independent of the fragment length. In the following, we show that even when these functions become length-dependent, $P(x)$ remains algebraic in the small size limit.

A. Arbitrary fragmentation density $\rho(l)$

Consider a fragmentation process in which an interval is broken into two fragments of relative lengths l and $1-l$ with an arbitrary fragmentation density $\rho(l)$. This density satisfies the constraint $\int dl\, \rho(l) = 1$ following from normalization and the symmetry requirement $\rho(l) = \rho(1-l)$. The governing equation for the fragment length density reads

$$P(x) = 2q\rho(x) + 2p\int_x^1 \frac{dy}{y}\, \rho(y) P\left(\frac{x}{y}\right). \tag{7}$$

The Mellin transform (3) of the length density satisfies $M(s) = 2\mu(s)\,[q + pM(s)]$, and consequently

$$M(s) = \frac{2q}{\mu^{-1}(s) - 2p}, \tag{8}$$

where $\mu(s) \equiv \int dl\, l^{s-1}\rho(l)$ denotes the Mellin transform of the fragmentation density $\rho(l)$. The symmetry of the fragmentation density implies $\mu(2) = 1/2$ and therefore $M(2) = 1$, which confirms the conservation of length. The normalization condition implies $\mu(1) = 1$, which confirms that the average total number of fragments is given by $M(1) = \langle N\rangle$, in agreement with Eq. (1).

The Mellin transform shows that the fragment length density is scale free only when the fragmentation density is uniform. Nevertheless, the algebraic small-size behavior remains robust. Indeed, Eq. (8) suggests that the most important property of $M(s)$ is a simple pole whose location $s = \gamma$ is found from relation $2p\mu(\gamma) = 1$. This simple pole implies a power-law asymptotics of the fragment length density

$$P(x) \simeq Ax^{-\gamma} \tag{9}$$

as $x \to 0$. The exponent γ can be determined from the entire fragmentation density $\rho(x)$ via the relation

$$2p\int dl\, l^{\gamma-1}\rho(l) = 1. \tag{10}$$

The prefactor in Eq. (9) is $A \equiv [q\mu(\gamma)]/[p\mu'(\gamma)]$; it is simply the residue of the pole at $s = \gamma$.

At the critical point $p_c = 1/2$ one has $\gamma = 1$, independent of the fragmentation density $\rho(l)$. The relation (10) also shows that a generic behavior $\gamma \to 2$ occurs if the probability of becoming stable vanishes, i.e., if $p \to 1$. In the complementary $p \to 0$ limit, the small-size behavior of $\rho(l)$ determines the small fragment distribution. In particular, if $\rho(l) \sim l^{-r}$ in the limit $l \to 0$, Eq. (10) shows that $\gamma \to r$ as $p \to 0$. Hence, in this case the restricted exponent range $r < \gamma < 2$ emerges by tuning p.

As an illustration, consider the fragmentation density $\rho(l) = B[l(1-l)]^{\delta-1}$, with $B = \Gamma(2\delta)/\Gamma^2(\delta)$ ensuring proper normalization. The exponent γ is determined from Eq. (10) to give

$$2p\,\frac{\Gamma(2\delta)\,\Gamma(\gamma+\delta-1)}{\Gamma(\delta)\,\Gamma(\gamma+2\delta-1)} = 1. \tag{11}$$

This relation shows that the exponent γ always belongs to the range $1-\delta<\gamma<2$. In the extreme case of $\delta\to 0$, the decay exponent is concentrated near $\gamma=1$. Such universal $P(x)\sim x^{-1}$ behavior is empirically observed in DNA segmentation algorithms. In the other extreme $\delta\to\infty$, the exponent simplifies to $\gamma=1+\ln 2p$. Note also that explicit results for both $P(x)$ and the exponent γ can be obtained for integer δ's. The case of $\delta=1$ corresponds to the uniform density. For $\delta=2$, the Mellin transform reads $\mu(s)=6/(s+1)(s+2)$, and from (8) we find that the fragment density is a combination of two power laws:

$$P(x)=\frac{12q}{\sqrt{1+48p}}\left(x^{\frac{3-\sqrt{1+48p}}{2}}-x^{\frac{3+\sqrt{1+48p}}{2}}\right). \qquad (12)$$

Generally, the length density is a linear combination of δ power laws for all integer δ's.

B. Arbitrary fragmentation probability $p(x)$

We now discuss the complementary generalization, in which the probability $p(x)$ that a new fragment remains unstable depends on the fragment size x. This is relevant for impact fragmentation and DNA segmentation where fragments have an intrinsic size scale below which the fragmentation probability becomes negligible. For an arbitrary fragmentation probability $p(x)$ the governing equation reads

$$P(x)=2\left[1-p(x)+\int_x^1\frac{dy}{y}\,p(y)P\left(\frac{x}{y}\right)\right]. \qquad (13)$$

Consequently, the Mellin transform of the length density admits the general solution

$$M(s)=\frac{2}{s}\,\frac{1-s\sigma(s)}{1-2\sigma(s)} \qquad (14)$$

where $\sigma(s)\equiv\int dx\,x^{s-1}p(x)$ is the Mellin transform of the probability $p(x)$.

The small size tail is determined by the poles of $M(s)$. When the condition $2\sigma(0)>1$ is satisfied, $M(s)$ has a simple pole at $s=\gamma$, and therefore, the small size behavior remains algebraic as in Eq. (9). The corresponding exponent γ is determined from $2\sigma(\gamma)=1$, or explicitly

$$2\int dx\,x^{\gamma-1}\,p(x)=1, \qquad (15)$$

and the prefactor $A=[\gamma-2]/[2\gamma\sigma'(\gamma)]$ equals the residue at $s=\gamma$. Equation (15) is a transcendental equation, and the details of the function $p(x)$ in the entire range $0<x<1$ determine the exponent γ. This situation is reminiscent of the behavior found when the fragmentation probability along the interval was not uniform.

In the complementary case of $2\sigma(0)<1$, Eq. (14) shows that $M(s)$ has a simple pole at $s=0$. This implies that the length density is regular in the small size limit: $P(x)\to 2/[1-2\sigma(0)]$ as $x\to 0$. In the marginal case $2\sigma(0)=1$, we find $M(s)\sim s^{-2}$, which leads to a logarithmic divergence of the length density, i.e., $P(x)\sim\ln\frac{1}{x}$ as $x\to 0$.

As an illustration, consider a solvable example: the fragmentation probability $p(x)=x^{\lambda}$ with $\lambda>0$. In this case we have $\sigma(s)=1/(s+\lambda)$, and consequently $M(s)=2\lambda/[s(s+\lambda-2)]$. Inverting the Mellin transform $M(s)$ gives the fragment length density

$$P(x)=\begin{cases}\frac{2\lambda}{2-\lambda}\left(x^{-(2-\lambda)}-1\right) & 0<\lambda<2;\\ 4\ln\frac{1}{x} & \lambda=2;\\ \frac{2\lambda}{\lambda-2}\left(1-x^{\lambda-2}\right) & 2<\lambda.\end{cases} \qquad (16)$$

Again, the range $0<\gamma<2$ becomes accessible.

Hence, the algebraic small size divergence is robust as it extends to situations where either the fragmentation density or the fragmentation probability are size dependent. The entire form of these functions is needed to calculate the corresponding power-law exponent.

In the rest of this paper, we restrict ourselves to the basic model where the fragmentation density is uniform, $\rho(x)=1$, and the fragmentation probability $p(x)\equiv p$ is size independent.

III. THE MOMENTS

The fragment size distribution represents an average over infinitely many realizations of the stochastic fragmentation process, and hence, it does not characterize sample-to-sample fluctuations. In this section, we show that fluctuations do not vanish in the thermodynamic limit, and therefore, the process is non-self-averaging. We investigate sample-to-sample fluctuations by computing the moments Y_α defined by

$$Y_\alpha=\sum_i x_i^\alpha, \qquad (17)$$

where the sum runs over all fragments in a given realization. These moments have proved useful in a variety of contexts including spin glasses, random maps, and random walks [29, 30]. The fact that these moments have non-trivial probability distributions is a signature of lack of self-averaging.

Before we attempt to derive these probability distributions, we start with the simpler task of computing the expected values of the moments $\langle Y_\alpha\rangle$ and their correlations $\langle Y_\alpha Y_\beta\rangle$. For integer α, $\langle Y_\alpha\rangle$ is the probability that α randomly chosen points in the unit interval belong to the same fragment. Similarly, for integer α and integer β, $\langle Y_\alpha Y_\beta\rangle$ is the probability that among $\alpha+\beta$ points chosen at random, the first α points all lie on the same fragment, and the last β points all lie on another (possibly the same) fragment.

The expected value of Y_α satisfies

$$\langle Y_\alpha\rangle = (q + p\langle Y_\alpha\rangle)\int dy\,[y^\alpha + (1-y)^\alpha]\,. \qquad (18)$$

The q-term corresponds to the situation where a first generation fragment becomes stable while the second term describes the complementary situation. Equation (18) gives

$$\langle Y_\alpha\rangle = \begin{cases} 2q/(\alpha+1-2p) & \alpha > 2p-1; \\ \infty & \alpha \le 2p-1. \end{cases} \qquad (19)$$

Since the single point averages are simply the moments, $\langle Y_\alpha\rangle = M(\alpha+1)$, in agreement with Eq. (4). Higher-order averages cannot be computed from the fragment size density. However, one can obtain exact expressions for higher averages from relations similar in spirit to Eq. (18). For instance, $\langle Y_\alpha Y_\beta\rangle$ satisfies

$$\begin{aligned}\langle Y_\alpha Y_\beta\rangle \;&=\; 2(q + p\langle Y_\alpha Y_\beta\rangle)\int dy\, y^{\alpha+\beta} \qquad (20)\\ &+\; 2(q+p\langle Y_\alpha\rangle)(q+p\langle Y_\beta\rangle)\int dy\, y^\alpha (1-y)^\beta.\end{aligned}$$

The expected values $\langle Y_\alpha\rangle$ and $\langle Y_\beta\rangle$ are already known, and the correlation $\langle Y_\alpha Y_\beta\rangle$ is found from (20) to give

$$\langle Y_\alpha Y_\beta\rangle = 2\,\frac{q + C(\alpha,\beta)(q+p\langle Y_\alpha\rangle)(q+p\langle Y_\beta\rangle)}{\alpha+\beta+1-2p} \qquad (21)$$

if $\alpha,\beta,\alpha+\beta > 2p-1$, and $\langle Y_\alpha Y_\beta\rangle = \infty$ otherwise. Here we used the shorthand notation $C(\alpha,\beta) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}$.

Equation (21) shows that $\langle Y_\alpha Y_\beta\rangle \neq \langle Y_\alpha\rangle\langle Y_\beta\rangle$ and, in particular, $\langle Y_\alpha^2\rangle \neq \langle Y_\alpha\rangle^2$. Hence, fluctuations in Y_α do not vanish in the thermodynamic limit, which implies that the stochastic fragmentation process is non-self-averaging. This means that statistical properties obtained by averaging over all realizations are insufficient to probe sample-to-sample fluctuations. Lack of self-averaging was also found in fragmentation processes that exhibit a shattering transition [31–33].

It is possible to evaluate higher-order averages such as $\langle Y_\alpha^n\rangle$. However, even for small n these averages become quite cumbersome and not terribly illuminating. Instead, one might try to obtain the distribution, $Q_\alpha(Y_\alpha)$, of possible outcomes of the moments Y_α. Let us first consider the fragment number distribution $Q_0(N)$ (the zeroth moment equals the number of fragments, $Y_0 = N$), which can be determined analytically. The minimal number of fragments is produced when both of the first generation fragments are stable, and hence, $Q_0(2) = q^2$. Similarly for $N \ge 3$ we obtain the recursion relation

$$\begin{aligned}Q_0(N) \;&=\; 2pqQ_0(N-1) \qquad (22)\\ &+\; p^2 \sum_{N_1+N_2=N} Q_0(N_1)Q_0(N_2),\end{aligned}$$

where the total number of fragments N is obtained in various ways from a smaller number of fragments that appear after fragmentation of the two first generation fragments. Specifically, if exactly one of the first generation fragments is unstable, it should produce $N-1$ stable fragments. If both of the first generation fragments are unstable, they can produce N_1 and N_2 fragments, respectively, subject to the constraint $N_1 + N_2 = N$. This explains the right-hand side of Eq. (22).

Equation (22) can be solved by introducing the generating function $\mathcal{Q}_0(z) \equiv \sum_{N\ge 2} Q_0(N) z^N$, which satisfies $p^2\mathcal{Q}_0^2(z) - (1-2pqz)\mathcal{Q}_0(z) + q^2z^2 = 0$. Solving this quadratic equation yields the generating function

$$\mathcal{Q}_0(z) = \frac{1 - 2pqz - \sqrt{1-4pqz}}{2p^2}. \qquad (23)$$

Expanding $\mathcal{Q}_0(z)$ in powers of z gives

$$Q_0(N) = \frac{\Gamma\left(N-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(N+1)}\,\frac{(4pq)^N}{4p^2}. \qquad (24)$$

At the critical point, $p_c = 1/2$, the number distribution decays algebraically in the large N limit:

$$Q_0(N) \sim N^{-3/2}. \qquad (25)$$

In the vicinity of $p_c = 1/2$, the number distribution attains the scaling form

$$Q_0(N) \sim N^{-3/2}\exp\left[-4N(\Delta p)^2\right], \qquad (26)$$

where $\Delta p = p_c - p$. Hence, below the critical point, the tail of the number distribution is exponential.

The probability that an infinite number of fragments is produced is given by

$$Q_0(\infty) = 1 - \sum_{N=2}^{\infty} Q_0(N) = 1 - \mathcal{Q}_0(z=1). \qquad (27)$$

Below the critical point, the number of fragments remains finite, i.e., $Q_0(\infty) = 0$. In the complementary case of $p > p_c$, with finite probability, an infinite number of fragments is produced

$$Q_0(\infty) = 1 - \frac{q^2}{p^2}. \qquad (28)$$

The case $\alpha = 0$ is unique in the sense that the variable $Y_0 = N$ is discrete. Another special case is $\alpha = 1$ when length conservation dictates $Y_1 = 1$, and therefore the distribution is trivial: $Q_1(Y_1) = \delta(Y_1 - 1)$. Generally, the distribution $Q_\alpha(Y_\alpha)$ is highly non-trivial. In the range $0 < \alpha < 1$ we of course have $Y_\alpha > 1$, i.e., Q_α has support in the interval $(1,\infty)$, as in the case of $\alpha = 0$. Equation (28) then suggests that for $p > p_c$ the distribution $Q_\alpha(Y_\alpha)$ should have a singular component at $Y_\alpha = \infty$.

In the following, we focus on the more interesting case of $\alpha > 1$. Here, the inequality $Y_\alpha < Y_1 = 1$ implies that the distribution $Q_\alpha(Y_\alpha)$ has support on the interval

$(0,1)$. If both of the first generation fragments happen to be stable, then $Y_\alpha = x^\alpha + (1-x)^\alpha$, where x is chosen uniformly in the unit interval. Both fragments are stable with probability q^2, and the corresponding contribution to $Q_\alpha(Y_\alpha)$, which we denote by $\Pi_\alpha(Y_\alpha)$, reads $\Pi_\alpha(Y_\alpha) = 2q^2 \frac{dx}{dY_\alpha}$, where x is the greater of the two roots of equation $Y_\alpha = x^\alpha + (1-x)^\alpha$ (the factor 2 accounts for the smaller root). Although it is generally impossible to express the above formula solely in terms of Y_α, in some special cases one can determine $\Pi_\alpha(Y_\alpha)$ explicitly, e.g., $\Pi_2(Y_2) = q^2(2Y_2-1)^{-1/2}$, $\Pi_3(Y_3) = q^2(3Y_3 - 3/4)^{-1/2}$; generally, $\Pi_\alpha(Y_\alpha) \sim (Y_\alpha - 2^{1-\alpha})^{-1/2}$.

Note that the distribution $\Pi_2(Y_2)$ has a singularity at $Y_2 = 1/2$, which obviously implies a singularity of $Q_2(Y_2)$ at the same point. To understand the origin of this singularity, notice that when the process ends with two stable fragments, then $Y_2 = x^2 + (1-x)^2 \geq 1/2$. Therefore, the behavior of $Q_2(Y_2)$ for the case of $Y_2 < 1/2$ is *not* affected by realizations with two final fragments, and this explains the singularity at $Y_2 = 1/2$. If the process ends with three stable fragments, then $Y_2 = x_1^2 + x_2^2 + (1-x_1-x_2)^2 \geq 1/3$. Similarly, if the process ends with k stable fragments, then $Y_2 \geq 1/k$. Hence, we anticipate that the distribution $Q_2(Y_2)$ has singularities at $Y_2 = 1/k$ for integer $k \geq 2$. Similar singularities underlie distributions of moments in a number of random processes, including random walks, spin glasses, random maps, and random trees [7, 8, 29, 30].

A straightforward generalization of the above argument suggests that for arbitrary $\alpha > 1$, the distribution $Q_\alpha(Y_\alpha)$ possesses singularities at $Y_\alpha = k^{1-\alpha}$. The existence of these infinitely many singularities shows that analytical determination of the distribution $Q_\alpha(Y_\alpha)$ is hardly possible. Indeed, $Q_\alpha(Y_\alpha)$ satisfies the difficult integral equation

$$\begin{aligned} Q_\alpha(Y_\alpha) &= \Pi_\alpha(Y_\alpha) \\ &+ 2pq \int \frac{dl}{(1-l)^\alpha}\, Q_\alpha\left(\frac{Y_\alpha - l^\alpha}{(1-l)^\alpha}\right) \\ +p^2 \int_0^{Y_\alpha} dZ & \int \frac{dl}{l^\alpha (1-l)^\alpha}\, Q_\alpha\left(\frac{Z}{l^\alpha}\right) Q_\alpha\left(\frac{Y_\alpha - Z}{(1-l)^\alpha}\right). \end{aligned} \tag{29}$$

Equation (29) has been derived by repeating the steps used in the derivation of (22). The first (second) term on the right-hand side of Eq. (29) corresponds to the case where two (one) of the first generation fragments are stable. The third convolution term describes the alternative case when both of the first generation fragments are unstable. Note that in addition to the recursive nature of the process, we have employed extensivity, i.e., $\langle Y_\alpha \rangle \propto l^\alpha$, in an interval of length l.

In order to study the small-Y_α behavior of the distribution, we employ the Laplace transform method. From Eq. (29), $R_\alpha(\lambda) \equiv \int_0^1 dY_\alpha\, e^{-\lambda Y_\alpha} Q_\alpha(Y_\alpha)$ obeys

$$R_\alpha(\lambda) = p^2 \int_0^1 dl\, R_\alpha\left[\lambda l^\alpha\right] R_\alpha\left[\lambda (1-l)^\alpha\right] + \ldots . \tag{30}$$

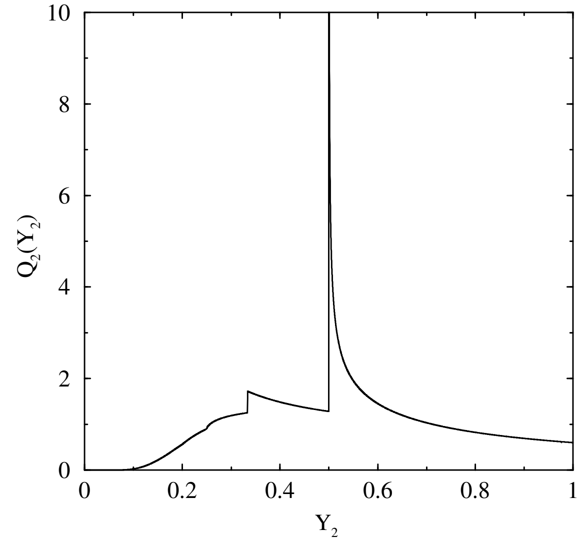

FIG. 1: The distribution of the second moment, $Q_2(Y_2)$ versus $Y_2 \equiv \sum_i x_i^2$, from numerical simulations with $p = 0.4$.

In Eq. (30), we do not write explicitly the Laplace transform of the first two terms of Eq. (29), because these two terms become negligible when $Y_\alpha \to 0$. The $Y_\alpha \to 0$ asymptotics of $Q_\alpha(Y_\alpha)$ is reflected by the $\lambda \to \infty$ asymptotics of $R_\alpha(\lambda)$. We argue that $R_\alpha(\lambda) \sim \exp(-A\lambda^\omega)$ with $\omega = 1/\alpha$. Assuming that $\alpha\omega > 1$, the above stretched exponential form of $R_\alpha(\lambda)$ shows that the product $R_\alpha\left[\lambda l^\alpha\right] R_\alpha\left[\lambda(1-l)^\alpha\right]$ would reach a maximum that greatly exceeds $R_\alpha(\lambda)$ at $l = 1/2$, in contradiction with Eq. (30). Alternatively, if $\alpha\omega < 1$, the above product would reach its maximum at $l = 0$ and $l = 1$. Then, the integral on the right-hand side of Eq. (30) would become $2R_\alpha(\lambda) \int dl\, R_\alpha\left[\lambda l^\alpha\right] \propto \lambda^{-1/\alpha} R_\alpha(\lambda)$, in contradiction with Eq. (30). Therefore, we conclude that $R_\alpha(\lambda) \sim \exp\left(-A\lambda^{1/\alpha}\right)$ as $\lambda \to \infty$. This behavior implies that the distribution $Q_\alpha(Y_\alpha)$ vanishes according to

$$Q_\alpha(Y_\alpha) \sim \exp\left(-B Y_\alpha^{-\frac{1}{\alpha-1}}\right) \tag{31}$$

as $Y_\alpha \to 0$. Therefore, the distribution $Q_\alpha(Y_\alpha)$ has an essential singularity at the origin, which completes a countable set of algebraic singularities located at $Y_\alpha = k^{1-\alpha}$ with $k = 2, 3, \ldots$

We performed Monte Carlo simulations of the stochastic fragmentation process. Hereinafter, we present simulation results for a representative case of $p = 0.4$. The data corresponds to an average over 5×10^{12} realizations. Figure 1 shows the probability distribution of the second moment. The distribution exhibits pronounced singularities at $Y_2 = 1/2$ and $Y_2 = 1/3$, while the following singularities are less visible. One can verify the existence of further singularities by differentiating $Q_2(Y_2)$. Figure 2 displays the essential singularity at the origin.

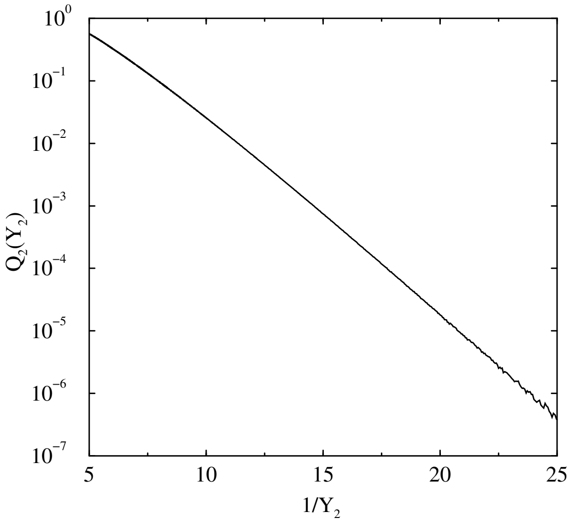

FIG. 2: The small size tail of the distribution of the second moment. Shown is $Q_2(Y_2)$ versus $1/Y_2$.

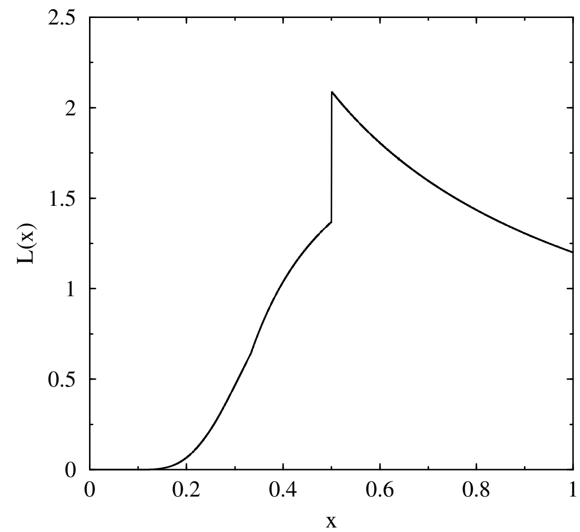

FIG. 3: The size distribution of the longest fragment.

IV. EXTREMAL CHARACTERISTICS

Extremal properties can be viewed as an additional probe of sample-to-sample fluctuations. Moreover, they are interesting on their own as they arise in many problems of mathematics, physics, and computer science [34–40]. The largest fragment is an important extremal characteristic. Obviously, when $x \geq 1/2$, the size distribution $L(x)$ of the largest fragment equals the length density, $L(x) = P(x) = 2q\,x^{-2p}$. In the complementary case of $x < 1/2$, $L(x)$ satisfies

$$\begin{aligned} L(x) &= 2qpL_-\left(\frac{x}{1-x}\right) + 2p\int_{1-x}^{1} \frac{dy}{y}\, L\left(\frac{x}{y}\right) \\ &+ 2p^2 \int_{x}^{1-x} \frac{dy}{y}\, L\left(\frac{x}{y}\right) L_-\left(\frac{x}{1-y}\right), \end{aligned} \tag{32}$$

where $L_-(u) \equiv \int_0^u dv\, L(v)$. The first term on the right-hand side of Eq. (32) describes the situation where the unit interval is fragmented into two intervals of lengths x and $1-x$, and where the smaller fragment is stable and the larger fragment is unstable (hence the factor qp). The latter L_- factor guarantees that subsequent fragmentation of the unstable interval does not lead to a longer fragment. If one of the first generation fragments is shorter than x, then only the longest first generation fragment contributes, which leads to the second term on the right-hand side of Eq. (32). The next term describes the situation where both of the first generation fragments are longer than x, so the longest fragment can result from breaking any of the two fragments. The factor L_- guarantees that the longest fragment of length x comes from the corresponding first generation fragment, and the factor p^2 guarantees that both of the first generation fragments are unstable.

Figure 3 shows that $L(x)$ is discontinuous at $x = 1/2$. This discontinuity can be understood by noting that $L(x)$ obeys different equations for $x > 1/2$ and $x < 1/2$, and hence it loses analyticity at the boundary. One can construct the entire solution recursively. We already know that $L(x) = 2q\,x^{-2p}$ for $x > 1/2$. Inserting this result into the right-hand side of Eq. (32) leads to an integral equation for $L(x)$ in the interval $1/3 < x < 1/2$. Solving that equation one finds $L(x)$. We do not quote the cumbersome solution and only note that the amplitude of the discontinuity is $2q^2$. Then we can use the solution for $x > 1/3$ to determine $L(x)$ in the interval $1/4 < x < 1/3$, and so on. Hence, $L(x)$ should possess an infinite set of singularities at $x = 1/k$, which become weaker as k increases. One can also understand why $L(x)$ is discontinuous at $x = 1/2$ by considering the $p = 0$ case where the distribution becomes a step function $L(x) = 2\theta(x - \frac{1}{2})$.

Consider now the complementary extremal characteristic — the shortest-segment size distribution $S(x)$. Clearly, $S(x) = 0$ for $x > 1/2$. If $x < 1/2$, one easily finds $S(x) = 2\theta(\frac{1}{2} - x)$ in the special case of $p = 0$. To proceed in the general case, we first note that, if the unit interval is divided into N fragments, the shortest fragment must obey $x_{\min} \leq 1/N$. Hence $1/3 < x < 1/2$ implies that the unit interval has been divided just once, i.e., both of the first generation fragments are stable. This shows that $S(x) = 2q^2$ when $1/3 < x < 1/2$. Finally, for $x < 1/3$ the shortest size distribution $S(x)$ obeys

$$\begin{aligned} S(x) &= 2q^2 + 2qpS_+\left(\frac{x}{1-x}\right) \\ &+ 2p\int_{x}^{1-x} \frac{dy}{y}\, S\left(\frac{x}{y}\right)\left[q + pS_+\left(\frac{x}{1-y}\right)\right], \end{aligned} \tag{33}$$

where $S_+(u) = \int_u^1 dv\, S(v)$. The first term on the right-

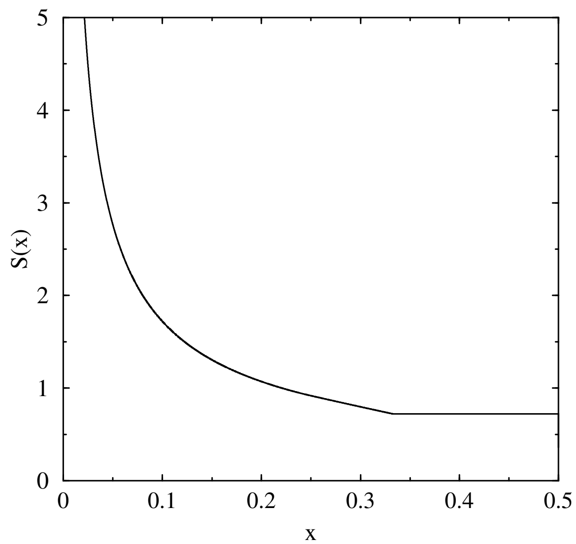

FIG. 4: The size distribution of the shortest fragment.

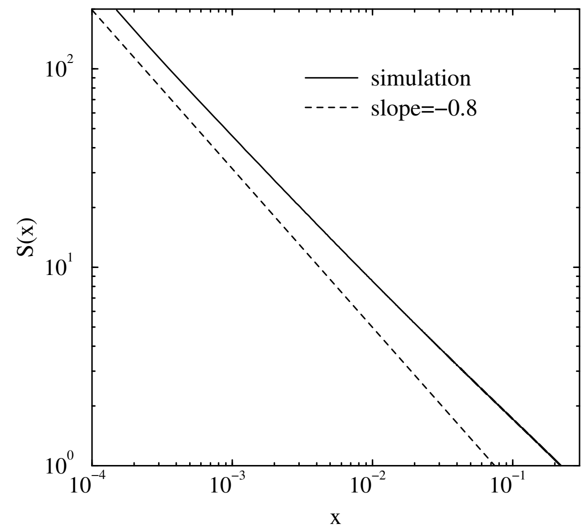

FIG. 5: The small size tail of the size distribution of the shortest fragment for $p = 0.4$. A line of slope -0.8 is shown for reference.

hand side of Eq. (33) describes the situation where both of the first generation fragments are stable. The second term corresponds to the case where the smaller first generation fragment of length x is stable while the longer fragment is unstable, with the S_+ factor ensuring that subsequent fragmentation of this longer fragment does not produce a fragment shorter than x. The last term describes various situations that are possible if both of the first generation fragments are longer than x.

We cannot obtain an analytical expression for $S(x)$ over the entire length range, because in every interval $(\frac{1}{k}, \frac{1}{k+1})$ a different analytical expression holds. In principle, however, one could determine $S(x)$ recursively. For instance, we already know $S(x)$ in the first two regions. Inserting those expressions into Eq. (33) yields

$$S(x) = 2q^2 + 4pq^3 \left(\frac{1}{2} - \frac{x}{1-x} + \ln \frac{1-x}{2x} \right) \quad (34)$$

in the third region $1/4 < x < 1/3$. Clearly, $S(x)$ possesses an infinite set of singularities at $x = 1/k$.

Figure 4 shows $S(x)$ for $x < 1/2$ and $p = 0.4$. One can see the plateau region $1/3 < x < 1/2$, and the value of $S(x)$ in this region agrees with the theoretical prediction $S(x) = 2q^2$. The divergence in the small size limit is consistent with the power law behavior: $S(x) \sim x^{-\delta}$ as $x \to 0$. Inserting this guess into Eq. (33) leads to a self-consistent value of the exponent $\delta = 2p$. Numerical simulations show that $S(x)$ slowly approaches the predicted behavior for the case $p = 0.4$ (see Fig. 5).

In deriving the relation $\delta = 2p$, we have implicitly assumed that the shortest-segment size distribution is nonsingular. This is indeed the case when $p \le p_c$. For $p > p_c$, however, the distribution $S(x)$ should additionally contain the singular component

$$S_{\rm sing}(x) = \Delta \delta(x) \quad (35)$$

with $\Delta = 1 - (q/p)^2$, reflecting that with finite probability, the total number of fragments is infinite, see Eq. (28).

A more direct way to derive the same result is to note that Δ, the probability that $x_{\rm min} = 0$, satisfies the relation $\Delta = 2pq\Delta + p^2[1 - (1-\Delta)^2]$. Indeed, the first term describes the situation when exactly one first generation fragment is unstable while the second term describes the situation when both of the first generation fragments are unstable. By solving the above equation we find two solutions, $\Delta = 0$ and $\Delta = (2p-1)/p^2$. The first solution applies when $p < p_c$; the second solution applies when $p > p_c$ and agrees with Eq. (35). In order to investigate the small-size asymptotics of $S(x)$, we write $S(x) = \Delta\delta(x) + \mathcal{S}(x)$ and assume that the continuous part follows the power law behavior, $\mathcal{S}(x) \sim x^{-\delta}$ as $x \to 0$. Substituting this into Eq. (33) and balancing the dominant terms yields $\delta = 2q$. To summarize, different behaviors characterize the small size tail of the length distribution of the shortest segment:

$$S(x) \sim \begin{cases} x^{-2p} & p < 1/2; \\ x^{-2q} & p > 1/2. \end{cases} \quad (36)$$

V. APPLICATION TO RANDOM SEQUENTIAL ADSORPTION

Random sequential adsorption (RSA) processes [41] have been applied to a wide range of chemical, biological, and physical processes. Examples include binding of proteins to surfaces [42, 43], genome sequencing [12–14], and granular compaction [44, 45]. In one dimension, k-mers are deposited (with a uniform rate) onto a linear lattice, and the deposition events are successful only when all k

sites are empty. Eventually, the system reaches a jamming state where no further deposition events are possible. Basic quantities of interest are the jamming density, ρ_∞, and its dependence on the initial concentration ρ_0, as well as the gap size distribution.

Adsorption can be viewed as a fragmentation process, and the above stochastic fragmentation process generalizes RSA to situations where the gaps between the adsorbed particles may become passive with probability p after each deposition event. Stable gaps can no longer be deposited onto, or in other words, fragmented. In the following, we study this RSA problem both in discrete and continuous space. We show that the limiting density of passive gaps is proportional to the fragment length density obtained above. In the interesting limit of vanishing initial concentrations, $\rho_0 \to 0$, the final jamming density vanishes according to $\rho_\infty \sim \rho_0^{2q}$ when $q < 1/2$, and as $\rho_\infty \sim \rho_0$ when $q > 1/2$. Hence, the jamming density is significantly enhanced over the initial density only if p exceeds the critical value $p_c = 1/2$.

A. Discrete Space

Let us consider random sequential adsorption in one spatial dimension were each deposition event creates two new smaller gaps. We assume that each of these gaps remains active with probability p, while with probability $q = 1 - p$ it becomes passive, i.e., adsorption events no longer occur on this gap. Initially, the system consists of randomly distributed monomers with density ρ_0, and all gaps are active. Then, r-mers are deposited with a temporally constant and spatially homogeneous rate, set to unity without loss of generality. The deposition events are successful if and only if all r sites are empty. Below, we consider the monomer case ($r = 1$).

Let $A_k(t)$ be the probability of finding an active gap of size k at time t. This distribution changes according to

$$\frac{dA_k(t)}{dt} = 2p \sum_{m=k+1}^{\infty} A_m(t) - kA_k(t). \tag{37}$$

The loss term reflects that deposition in each of the empty sites destroys the gap, and the gain term reflects the fact that two smaller gaps are created in each deposition event. The prefactor of the gain term is equal to the production rate. Similarly, the probability $P_k(t)$ of finding a passive gap of size k satisfies

$$\frac{dP_k(t)}{dt} = 2q \sum_{m=k+1}^{\infty} A_m(t). \tag{38}$$

The evolution equations (37)–(38) conserve the total length $\sum_k (k+1)(A_k + P_k) = 1$.

Let us consider systems that initially consist of randomly distributed monomers with all gaps being active, i.e., $A_k(0) = \rho_0^2(1-\rho_0)^k$ and $P_k(0) = 0$. Since P_k is enslaved to A_k, we derive the latter quantity first. The linear loss rate suggests the exponential ansatz

$$A_k = \alpha\beta^k \tag{39}$$

with the initial values $\alpha(0) = \rho_0^2$ and $\beta(0) = 1 - \rho_0$. Substituting (39) into (37) yields

$$\frac{d\beta}{dt} = -\beta, \qquad \frac{d}{dt}\ln\alpha = 2p\,\frac{\beta}{1-\beta}. \tag{40}$$

The first equation yields $\beta(t) = (1-\rho_0)e^{-t}$, and the second equation can be conveniently solved by changing the time variable from t to β using $\beta dt = -d\beta$. This transforms the second equation into $\frac{d}{d\beta}\ln\alpha = -2p(1-\beta)^{-1}$. Integrating this equation subject to the above initial condition yields $\alpha = \rho_0^{2q}(1-\beta)^{2p}$. The time-dependent gap distribution is therefore

$$A_k = \rho_0^{2q}(1-\beta)^{2p}\beta^k, \qquad \beta = (1-\rho_0)e^{-t}. \tag{41}$$

Next, we study the final jamming density which can be obtained from the active gap distribution by integration of the overall deposition rate over time, i.e., $\rho_\infty - \rho_0 = \int_0^\infty dt \sum_k kA_k(t) = \int_0^\infty dt\, \alpha\beta(1-\beta)^{-2}$. Again, it is useful to transform t to β. Evaluating the integral gives

$$\rho_\infty = \begin{cases} (\rho_0^{2q} - 2q\rho_0)/(1-2q) & q \neq 1/2; \\ \rho_0 \ln(1/\rho_0) & q = 1/2. \end{cases} \tag{42}$$

It is easy to verify that $\rho_\infty \to 1$ in the limit $q \to 0$. In the more interesting limit when the system is initially almost empty, i.e., when $\rho_0 \to 0$, the following leading behaviors emerge

$$\rho_\infty \sim \begin{cases} \rho_0^{2q} & q < 1/2; \\ \rho_0 \ln(1/\rho_0) & q = 1/2; \\ \rho_0 & q > 1/2. \end{cases} \tag{43}$$

In other words, the final coverage depends algebraically on the initial coverage in the range $q < 1/2$. In this case, the adsorption process can be viewed as "effective" since the increase in density is significant. Otherwise, the final density is proportional to the initial density ρ_0. The critical point is marked by a weak logarithmic increase in the jamming density.

We turn now to the distribution of passive gaps which is obtained by integrating (38) with $A_k(t)$ given by Eq. (41). In order to compare the gap-size distribution with the fragmentation case, we focus on the limiting ($t \to \infty$) distribution of passive gaps, which reads

$$P_k(\infty) = 2q\rho_0^{2q} \int_0^{1-\rho_0} d\beta\,(1-\beta)^{2p-1}\beta^k. \tag{44}$$

In the limit of almost empty initial conditions, $\rho_0 \to 0$, the average gap size diverges as $\langle k \rangle = (1-\rho_\infty)/\rho_\infty \sim \rho_\infty^{-1}$. Hence, the most interesting behavior emerges in the scaling region $\rho_0 \to 0$ and $k \to \infty$ with the scaling variable $\xi = k\rho_0$ kept fixed. In this region the limiting gap distribution (44) can be rewritten in the scaling form

$$P_k(\infty) = 2q\rho_0^{2q} k^{-2p}\Gamma(2p,\xi), \tag{45}$$

where $\Gamma(a,\xi) = \int_\xi^\infty dx\, x^{a-1}e^{-x}$ denotes the incomplete gamma function. Equation (45) shows that the gap size distribution behaves algebraically as long as the size of the gap does not exceed the average initial size $k^* = \rho_0^{-1}$, while for larger gaps the size distribution is suppressed exponentially

$$P_k(\infty) \simeq \begin{cases} 2q\Gamma(2p)\rho_0^{2q}k^{-2p} & k \ll k^*; \\ 2q\rho_0 k^{-1}e^{-k/k^*} & k \gg k^*. \end{cases} \tag{46}$$

These two expressions are indeed of the same order, $P \sim \rho_0^2$, in the vicinity of the crossover point $k \sim k^*$. In general, we find the algebraic behavior $P_k(\infty) \sim k^{-2p}$ in the limit $\rho_0 \to 0$, in agreement with the stable fragment length density of Eq. (5). This agrees with intuition since in the limit $\rho_0 \to 0$, stochastic RSA is a discrete counterpart of the stochastic fragmentation.

The above treatment can be generalized to the dimer ($r = 2$) case and even to the general r-mer case. Although these solutions become very cumbersome as r grows, the asymptotic behavior found for the monomer case including the scaling form of $P_k(\infty)$ and the jamming density ρ_∞ are not altered.

B. Continuous Space

The continuum limit where particles of unit length are deposited irreversibly onto a line can be obtained from the discrete r-mer case by taking the limit $r \to \infty$ and by redefining the time variable $rt \to t$ and the initial density $r\rho_0 \to \lambda$. Then, the densities of active and passive gaps of size x evolve according to

$$\begin{aligned} \frac{\partial A(x,t)}{\partial t} &= 2p\int_{x+1}^\infty dy\, A(y,t) - \theta(x-1)(x-1)A(x,t), \\ \frac{\partial P(x,t)}{\partial t} &= 2q\int_{x+1}^\infty dy\, A(y,t), \end{aligned} \tag{47}$$

where $\theta(x)$ is the step function. The initial conditions read $A(x,0) = \lambda^2 e^{-\lambda x}$ and $P(x,0) = 0$.

We first derive the distribution of active gaps of lengths $x \geq 1$. Equation (47) suggests that it remains exponential throughout the evolution, i.e.,

$$A(x,t) = \Phi(t)\exp\left[-\lambda x - (x-1)t\right]. \tag{48}$$

Substituting this exponential form into Eq. (47) yields $\frac{d}{dt}\ln\Phi(t) = 2p\, e^{-\lambda-t}/(\lambda+t)$. Integrating this differential equation subject to the initial conditions $\Phi(0) = \lambda^2$ gives the time dependent prefactor

$$\Phi(t) = \lambda^{2q}(\lambda+t)^{2p}\exp\left[-2p\int_\lambda^{\lambda+t} d\tau\,\frac{1-e^{-\tau}}{\tau}\right]. \tag{49}$$

The jamming density can be obtained by integrating the total deposition rate over time. We get

$$\rho_\infty = \int_0^\infty dt \int_1^\infty dx\,(x-1)P(x,t),$$

which simplifies to

$$\rho_\infty = \lambda^{2q}e^{-\lambda}\int_\lambda^\infty \frac{dt}{t^{2q}}\exp\left[-2p\int_\lambda^t d\tau\,\frac{1-e^{-\tau}}{\tau}\right]. \tag{50}$$

When $p = 1$, this expression agrees with the jamming density of the parking model with and without disorder [46, 47]. Independent of the probability p, the approach to the jamming state follows the classical t^{-1} law [41]

$$\rho_\infty - \rho(t) \simeq \lambda^{2q}e^{-\lambda}(\lambda+t)^{-1} \sim t^{-1}. \tag{51}$$

Additionally, the leading behavior in the limit of dilute initial conditions ($\lambda \to 0$) can be evaluated and the behavior found in Eq. (43) generalizes to the continuum limit

$$\rho_\infty \sim \begin{cases} \lambda^{2q} & q < 1/2; \\ \lambda\ln(1/\lambda) & q = 1/2; \\ \lambda & q > 1/2. \end{cases} \tag{52}$$

The limiting passive gap distribution is found by integrating the rate equation (47) using the active gap distribution

$$P_\infty(x) = 2q\int_0^\infty \frac{dt}{\lambda+t}\,\Phi(t)e^{-\lambda-(\lambda+t)x}, \tag{53}$$

with $\Phi(t)$ given by Eq. (49). In the limit of dilute initial conditions ($\lambda \to 0$) one can simplify the integral on the right-hand side of Eq. (53) to find the following extremal behaviors of the gap distribution

$$P_\infty(x) \simeq \begin{cases} 2q\lambda x^{-1}e^{-\lambda x} & x \gg \lambda^{-1}; \\ 2q\Gamma(2p)\lambda^{2q}x^{-2p} & 1 \ll x \ll \lambda^{-1}; \\ 2q\, e^{-2p\gamma_E}\lambda^{2q}\,\ln\left(\frac{1}{\lambda x}\right) & x \ll 1; \end{cases} \tag{54}$$

where $\gamma_E = 0.577215\ldots$ denotes the Euler constant. The first two asymptotics in (54) are straightforward extensions of the corresponding behaviors in the lattice case; the last line in (54) has been derived from (53) using the asymptotic relation

$$\int_0^T d\tau\,(1-e^{-\tau})/\tau = \ln T + \gamma_E + \mathcal{O}\left(e^{-T}\right).$$

Finally, we note that even in the long-time limit the density of active gaps does *not* vanish for sufficiently short gaps, $x \leq 1$. One can determine $A_\infty(x)$, and more generally $A(x,t)$, by employing an elementary relation between the densities of active and passive gaps, namely

$$A(x,t) = A(x,0) + \frac{p}{q}\,P(x,t). \tag{55}$$

This relation immediately follows from the master equations (47), and it clearly holds for arbitrary t as long as $x \leq 1$. By combining (54) and (55) we find

$$A_\infty(x) \simeq 2p\, e^{-2p\gamma_E}\, \lambda^{2q}\, \ln\left(\frac{1}{\lambda x}\right), \tag{56}$$

which applies if $\lambda \ll 1$ and $x \ll 1$.

Therefore, stochastic fragmentation processes can be naturally extended to adsorption processes, and apart from numeric prefactors the algebraic fragment-size distribution is reproduced in the limit of empty initial conditions. The phase transition underlying the branching process has an interesting implication. The jamming density is significantly larger than the initial density only when $p > 1/2$. The super-critical nature of the underlying branching process allows for an infinite number of fragments produced from a single fragment, and this explains the enhanced jamming density in the stochastic RSA process.

Although the fragmentation and the adsorption results are closely related, we have used two complementary approaches to obtain them. In the former case, it was convenient to bypass the distribution of unstable fragments and solve directly for the final stable fragment distribution, while in the latter case, it was more natural to study the entire time dependent behavior of both distributions. A more complete treatment of the fragmentation process is of course possible using a continuous time formulation which utilizes rate equations similar to (47).

VI. SUMMARY

We have studied a class of stochastic fragmentation processes, where fragments may become stable ("frozen") after each fragmentation event. We have found that in general, these processes are characterized by an algebraic small-size divergence of the fragment size distribution. This behavior is robust as it holds for size dependent fragmentation densities and fragmentation probabilities, as well as in dual adsorption processes in both continuous and discrete space. The corresponding power-law exponents can be tuned by varying the fragmentation probability, and the entire range allowed by mass conservation may be realized.

While the size density can be determined analytically, additional statistical measures of fluctuations are more difficult to handle. Nevertheless, we have shown that moments of the distribution exhibit large sample-to-sample fluctuations, and hence, knowledge of the entire distribution of observables is needed to characterize the system. Additionally, the distribution of the moments and of extremal characteristics, such as the longest and the shortest fragments, possesses an infinite set of singularities.

Lack of self-averaging is important in practical applications such as utilization of DNA segmentation for comparison of genomes of different species. Great care is clearly needed in comparative analysis of the segment length distributions as observed deviations between segments may be actually statistical rather than biological.

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