Stationary states and energy cascades in inelastic gases

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We find a general class of nontrivial stationary states in inelastic gases where, due to dissipation, energy is transfered from large velocity scales to small velocity scales. These steady-states exist for arbitrary collision rules and arbitrary dimension. Their signature is a stationary velocity distribution $f(v)$ with an algebraic high-energy tail, $f(v) \sim v^{-\sigma}$. The exponent σ is obtained analytically and it varies continuously with the spatial dimension, the homogeneity index characterizing the collision rate, and the restitution coefficient. We observe these stationary states in numerical simulations in which energy is injected into the system by infrequently boosting particles to high velocities. We propose that these states may be realized experimentally in driven granular systems.

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Energy dissipation has profound consequences in granular media. It is responsible for collective phenomena such as hydrodynamic instabilities [1, 2], shocks [3–5], and clustering [6–8]. The inelastic nature of the collisions remains crucially important in dilute settings and under vigorous forcing where, in contrast with molecular gases, there is no energy equipartition [9, 10] and the velocity distributions are typically non-Maxwellian [11–16]. In this letter, we show that energy dissipation also results in nonequilibrium stationary states where energy cascades from large velocity scales to small velocity scales, and we propose that these states may be realized experimentally in driven granular gases.

Hydrodynamic theory of granular flows is formulated using inelastic gases as a starting point [17–21]. Without forcing, dissipation is quantified via the energy balance equation, $dT/dt = -\Gamma$, where $T = \langle v^2\rangle$ is the granular temperature and Γ the dissipation rate. Collisions involving a pair of particles with relative velocity Δv occur with rate $\propto (\Delta v)^\lambda$ and result in energy loss $\propto (\Delta v)^2$. Assuming that a single velocity scale characterizes the system, $\Gamma \sim \langle(\Delta v)^{2+\lambda}\rangle \sim T^{1+\lambda/2}$, leads to Haff's cooling law, $T \sim t^{-2/\lambda}$ (exponential decay occurs when $\lambda = 0$). Hence, the temperature decays indefinitely and the velocity distribution $f(v)$ approaches the trivial steady-state $f(v) \to \delta(v)$ where all particles are at rest, a stationary state that can be considered as a dynamical fixed point. However, the energy balance equation assumes a finite dissipation rate. This need not be the case!

Our main result is that there is a general class of nontrivial stationary states where the velocity distribution has an algebraic high-energy tail

$$f(v) \sim v^{-\sigma} \tag{1}$$

as $v \to \infty$. First, we obtain this result in one-dimension and then, generalize it to arbitrary dimension.

Our system is a spatially homogeneous ensemble of identical particles that undergo inelastic collisions

$$v_{1,2} = pu_{1,2} + qu_{2,1} \tag{2}$$

with $u_{1,2}$ ($v_{1,2}$) the pre-collision (post-collision) velocities. The collision parameters p and q obey $p + q = 1$ so momentum is conserved. In each collision, the relative velocity is reduced by the restitution coefficient $r = 1 - 2p$ and the energy loss equals $pq(u_1 - u_2)^2$. The energy dissipation is maximal for completely inelastic collisions ($r = 0$, $p = 1/2$), and it vanishes for elastic collisions ($r = 1$, $p = 0$). We consider general collision rates $|u_1 - u_2|^\lambda$ with $0 \le \lambda \le 1$ the homogeneity index. Hard-spheres, $\lambda = 1$, model ordinary granular media. Maxwell molecules, where the collision rate is velocity-independent [22], $\lambda = 0$, model granular particles with dipole interactions, such as magnetic particles [23].

We seek stationary velocity distributions $f(v)$ that obey the Boltzmann equation, with the space and time derivatives set to zero,

$$\iint du_1\, du_2\, |u_1 - u_2|^\lambda f(u_1)f(u_2) \times [\delta(v - pu_1 - qu_2) - \delta(v - u_1)] = 0 \tag{3}$$

reflecting balance of gain and loss due to collisions.

The case $\lambda = 0$ is instructive as the velocity distribution can be obtained explicitly. The Boltzmann equation is a simple convolution because the collision rate is constant, and it is conveniently studied using the Fourier transform $F(k) = \int dv\, f(v)\, e^{ikv}$ that satisfies the nonlinear, nonlocal equation $F(k) = F(pk)F(qk)$. Normalization implies $F(0) = 1$. For elastic collisions ($p = 0$) any distribution is stationary, but this is just a one-dimensional pathology. Generally, for any value of the restitution coefficient, there is a family of solutions $F(k) = \exp\big(-v_0|k|\big)$ with arbitrary typical velocity v_0. As a result, the velocity distribution is Lorentzian

$$f(v) = \frac{1}{\pi v_0}\frac{1}{1 + (v/v_0)^2}. \tag{4}$$

We note that this distribution is independent of the restitution coefficient. The typical velocity v_0 is finite, but the average energy is infinite.

This distribution does not evolve under the inelastic collision dynamics. From the energy balance equation, one might have expected that the velocity distribution constantly narrows because particles dissipate energy. Surprisingly, there is perfect balance between collisional loss and gain at every velocity so that a steady velocity distribution is maintained. We conclude that in addition to the trivial state where all particles are at rest, there is another fixed point, a nonequilibrium stationary state.

For general collision rules, the characteristic exponent σ can be found analytically. For large v, the convolution in Eq. (3) is governed by the product $f(u_1)f(u_2)$ with one of the pre-collision velocities large and the other small because the distribution decays sharply at large velocities. The Boltzmann equation includes a loss term and a gain term. For the loss term, $u_1 = v$ is large and u_2 is small. For the gain term, there are two separate contributions. Either $v = pu_1$ and then u_2 is small or $v = qu_2$ and then u_1 is small. In either case, the double integral separates. The integral over the smaller velocity equals one. With the remaining integral over the larger velocity, denoted by u, the nonlinear Boltzmann equation becomes linear for large v

$$\int du\,|u|^\lambda f(u)\,[\delta(v-pu)+\delta(v-qu)-\delta(v-u)] = 0. \qquad (5)$$

Consequently, the tail of the distribution satisfies the functional equation

$$\frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right) - f(v) = 0 \qquad (6)$$

describing cascade of energy from large velocities to small ones, $v \to (pv, qv)$. The power-law decay (1) satisfies this equation when $p^{\sigma-1-\lambda} + q^{\sigma-1-\lambda} = 1$, and since $p + q = 1$, the characteristic exponent is

$$\sigma = 2 + \lambda. \qquad (7)$$

In one-dimension, σ is independent of the restitution coefficient.

We comment that the full nonlinear Boltzmann equation assumes molecular chaos as the two-particle velocity distribution is a product of one-particle distributions. However, the equation governing the tail is linear and it is valid under less stringent conditions. The only requirement is that energetic particles are weakly correlated with slower ones.

In arbitrary dimension d, the collision law is

$$\mathbf{v}_1 = \mathbf{u}_1 - (1-p)(\mathbf{u}_1 - \mathbf{u}_2)\cdot\hat{\mathbf{n}}\,\hat{\mathbf{n}}. \qquad (8)$$

Momentum transfer occurs only along the normal direction with $\hat{\mathbf{n}}$ a unit vector parallel to the impact direction connecting the particles. The energy dissipation equals $p(1-p)|(\mathbf{u}_1 - \mathbf{u}_2)\cdot\hat{\mathbf{n}}|^2$ and the collision rate is $|(\mathbf{u}_1 - \mathbf{u}_2)\cdot\hat{\mathbf{n}}|^\lambda$.

We derive the linear equation governing the tail of the distribution directly from the collision rule. With the large velocity $\mathbf{u}$, its magnitude $u \equiv |\mathbf{u}|$ and $\mu = (\hat{\mathbf{u}}\cdot\hat{\mathbf{n}})^2$, the collision rate is $u^\lambda \mu^{\lambda/2}$. The cascade process is $v \to (\alpha v, \beta v)$ with the stretching parameters $\alpha = (1-p)\mu^{1/2}$ and $\beta = [1-(1-p^2)\mu]^{1/2}$ for $\mathbf{v} = (1-p)\mathbf{u}\cdot\hat{\mathbf{n}}\,\hat{\mathbf{n}}$ and $\mathbf{v} = \mathbf{u} - (1-p)\mathbf{u}\cdot\hat{\mathbf{n}}\,\hat{\mathbf{n}}$, respectively [24]. Integrating over the impact angle $\hat{\mathbf{n}}$ and the velocity magnitude u, Eq. (5) generalizes as follows

$$\left\langle \int du\, u^{\lambda+d-1} f(u)[\delta(v-\alpha u)+\delta(v-\beta u)-f(v)]\mu^{\lambda/2}\right\rangle = 0.$$

Here, the brackets are used as shorthand for the angular integration, $\langle g\rangle = \int d\hat{\mathbf{n}}\, g(\hat{\mathbf{n}})$. Integrating over u yields

$$\left\langle \left[\frac{1}{\alpha^{d+\lambda}} f\left(\frac{v}{\alpha}\right) + \frac{1}{\beta^{d+\lambda}} f\left(\frac{v}{\beta}\right) - f(v)\right]\mu^{\lambda/2}\right\rangle = 0. \qquad (9)$$

The power-law decay (1) satisfies this equation provided that the exponent σ is root of the equation

$$\left\langle \left(\alpha^{\sigma-d-\lambda} + \beta^{\sigma-d-\lambda} - 1\right)\mu^{\lambda/2}\right\rangle = 0. \qquad (10)$$

The angular integration is performed using spherical coordinates. Given $\mu = \cos^2\theta$, then $d\hat{\mathbf{n}} \propto (\sin\theta)^{d-2} d\theta$, and $\langle\, g\,\rangle \propto \int_0^1 d\mu\, \mu^{-\frac{1}{2}}(1-\mu)^{\frac{d-3}{2}} g(\mu)$. The equality for the exponent σ can be written in terms of the gamma function and the hypergeometric function [25]

$$\frac{1 - {}_2F_1\left(\frac{d+\lambda-\sigma}{2}, \frac{\lambda+1}{2}, \frac{d+\lambda}{2}, 1-p^2\right)}{(1-p)^{\sigma-d-\lambda}} = \frac{\Gamma(\frac{\sigma-d+1}{2})\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{\sigma}{2})\Gamma(\frac{\lambda+1}{2})}. \qquad (11)$$

In the cascade process $v \to (\alpha v, \beta v)$, the total velocity magnitude increases $(\alpha + \beta \geq 1)$ even though the total energy decreases $(\alpha^2 + \beta^2 \leq 1)$. From these inequalities, the left-hand-side of (10) is positive when $\sigma - d - \lambda = 1$ but negative when $\sigma - d - \lambda = 2$. This gives the bounds $1 \leq \sigma - d - \lambda \leq 2$. The lower bound (7) is realized in one-dimension where collisions are head-on ($\mu = 1$). The upper bound is approached, $\sigma \to d + 2 + \lambda$, in the quasi-elastic limit, $r \to 1$. Clearly, dissipation is responsible for these steady states. For elastic collisions, the velocity distributions are Maxwellian, so the no-dissipation limit is singular. Interestingly, the energy may be either finite or infinite, depending on whether σ is larger or smaller than $d + 2$ [26]. In either case, the integral underlying the dissipation rate is divergent, a general characteristic of the stationary distributions.

Generally, the exponent σ increases monotonically with the spatial dimension d, the homogeneity index λ, and the restitution coefficient r. For completely inelastic ($r = 0$) hard spheres, we find $\sigma = 4.14922$ and $\sigma = 5.23365$ for $d = 2, 3$, respectively. The exponents for completely inelastic Maxwell molecules are $\sigma = 3.19520$ and $\sigma = 4.28807$ for $d = 2, 3$. These values provide lower bounds on σ with respect to r, as shown in Fig. 1.

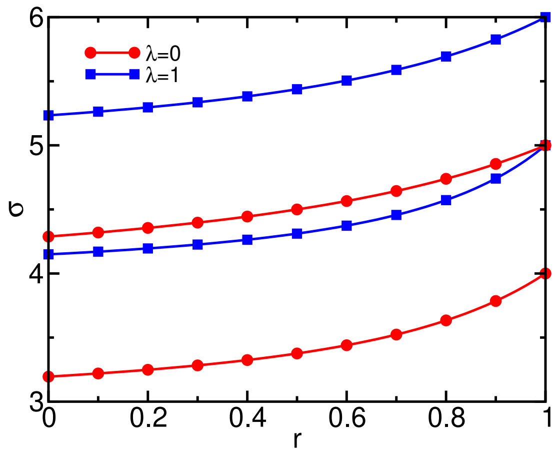

FIG. 1: The exponent σ as a function of the restitution coefficient r for $\lambda = 0$ (circles) and $\lambda = 1$ (squares). The lower curves correspond to $d = 2$ and the upper to $d = 3$.

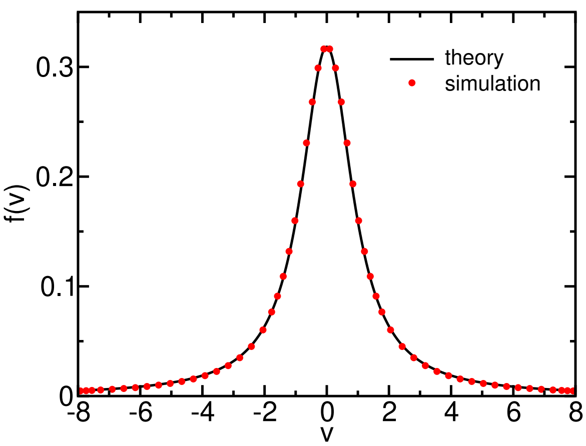

FIG. 2: The velocity distribution $f(v)$ versus v for Maxwell molecules in one-dimension. The ratio between the typical injection energy V and typical velocity v_0 is $V/v_0 \approx 10^4$.

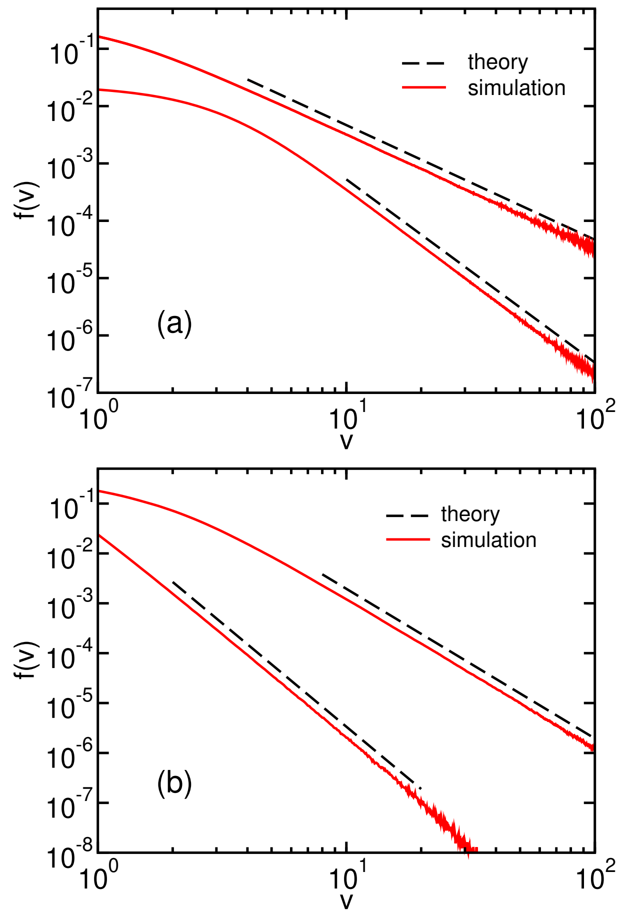

FIG. 3: The velocity distribution $f(v)$ versus v for Maxwell molecules (a) and hard spheres (b) in one-dimension (upper lines) and two-dimension (lower lines). The theory is shown for reference and $V/v_0 \approx 10^4$, 10^2 for (a), (b) respectively.

A remarkable feature is that the characteristic exponent changes continuously with the parameters d, λ, and r. Hence, the tails are not universal. The power-law decay stands in contrast with the stretched exponentials, $f(v) \sim \exp(-v^\nu)$, with $\nu = \lambda$ and $\nu = 1 + \lambda/2$, respectively, for unforced and thermally forced inelastic gases [12–15]. Previously, algebraic tails with different exponents were found but for non-stationary distributions describing freely cooling Maxwell molecules [27–29].

These steady-state distributions can be realized, up to some cutoff, in finite systems of driven inelastic particles. The key is to inject energy at a very large velocity scale. We used a Discrete Simulation Monte Carlo method in which pairs of particles are chosen at random with rate proportional to the collision rate and collided according to the collision rule (8). Initially, we start with an innocuous velocity distribution, uniform with support in $[-1 : 1]$. An energy loss "counter" keeps track of the aggregate energy loss. With rate γ, small compared with the collision rate, a particle is selected at random and it is energized by an amount equal to the aggregate energy loss. Subsequently, the energy loss counter is reset to zero. In this simulation, the total energy remains practically constant and more importantly, energy is injected only at the tail of the distribution. In effect, this procedure does not alter the purely dissipative dynamics, except for setting a scale for the most energetic particles.

We simulated completely inelastic hard-spheres and Maxwell molecules in one- and two-dimensions. We used $N = 10^7$ ($N = 10^5$) particles and injection rate $\gamma = 10^{-4}$ ($\gamma = 10^{-2}$) for $\lambda = 0$ ($\lambda = 1$). We verified that: (i) after a transient, the velocity distribution becomes stationary, (ii) the velocity distribution is Lorentzian for one-dimensional Maxwell molecules (Fig. 2), (iii) the velocity distribution has an algebraic tail, and (iv) σ is consistent with theory (Fig. 3).

The very same stationary distributions can be reached using other simulation methods. For example, particles may be re-energized such that their velocity is drawn from a Maxwellian distribution with a typical energy proportional to the system size (the data presented for one-dimensional hard-spheres are from such a simulation). We conclude that the simulations confirm the existence of the nontrivial stationary states with power-law tails. These steady-states are stable fixed points as the system is driven into them even when starting from compact distributions. Moreover, stability analysis of the time dependent version of Eq. (9) shows that the stationary distribution (1) is stable with respect to perturbations consisting of steeper algebraic tails.

Clearly, if $f(v)$ is a steady-state so is $v_0^{-d}f(v/v_0)$ for an arbitrary typical velocity v_0. The injection protocol selects v_0. Suppose that particles are boosted at rate γ per particle to velocity V, a scale that sets an upper cutoff on the velocity distribution [30]. The energy injection rate is γV^2, and the energy dissipation rate is $\Gamma \sim \langle v^{2+\lambda}\rangle$ with $\langle g\rangle \equiv \int^V dv\, v^{d-1} f(v)g(v)$. Energy balance, $\Gamma \sim \gamma V^2$, relates the injection rate, the injection scale, and the typical velocity scale,

$$\gamma \sim V^{\lambda}(V/v_0)^{d-\sigma}. \tag{12}$$

In our simulations, energy is maintained constant, $\langle v^2\rangle \sim 1$. When $\sigma < d+2$, the constant energy constraint implies $V^{d+2-\sigma} \sim v_0^{d-\sigma}$, that combined with energy balance (12) reveals how the maximal velocity, $V \sim \gamma^{-1/(2-\lambda)}$, and the typical velocity, $v_0 \sim \gamma^{\frac{d+2-\sigma}{(\sigma-d)(2-\lambda)}}$, scale with the injection rate. When $\sigma > d+2$, the initial conditions set the typical velocity: $\langle v^2\rangle \sim v_0^2$ implies $v_0 \sim 1$, and energy balance (12) yields $V \sim \gamma^{-1/(\sigma-d-\lambda)}$. The simulations are consistent with these scaling laws. For example, $V \approx 10^2$ and $v_0 \approx 10^{-2}$ for one-dimensional Maxwell-molecule simulations with $\gamma = 10^{-4}$. This, combined with the simulations, demonstrates that finite energy and infinite energy cases differ quantitatively but not qualitatively.

In conclusion, we find that the Boltzmann equation for homogeneous inelastic gases has nontrivial stationary solutions where the velocity distribution has an algebraic high-energy tail corresponding to an energy cascade from high to low velocities. These stationary states exist for arbitrary collision rates, arbitrary collision rules, and arbitrary spatial dimensions. Nonequilibrium steady states with power law tails over a finite range can be maintained by injecting energy at a large velocity scale. We have observed these steady states in numerical simulations and propose that they may be realized in granular gas experiments in which energy is added at large velocity scales via rare but powerful energy injections.

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