Kinetics of Clustering in Traffic Flows

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We study a simple aggregation model that mimics the clustering of traffic on a one-lane roadway. In this model, each "car" moves ballistically at its initial velocity until it overtakes the preceding car or cluster. After this encounter, the incident car assumes the velocity of the cluster which it has just joined. The properties of the initial distribution of velocities in the small velocity limit control the long-time properties of the aggregation process. For an initial velocity distribution with a power-law tail at small velocities, $P_0(v) \sim v^\mu$ as $v \to 0$, a simple scaling argument shows that the average cluster size grows as $n \sim t^{(\mu+1)/(\mu+2)}$ and that the average velocity decays as $v \sim t^{-1/(\mu+2)}$ as $t \to \infty$. We derive an analytical solution for the survival probability of a single car and an asymptotically exact expression for the joint mass-velocity distribution function. We also consider the properties of spatially heterogeneous traffic and the kinetics of traffic clustering in the presence of an input of cars.

I. INTRODUCTION

A variety of approaches have been applied to describe the collective properties of traffic flows [1]. For example, to mimic congested traffic flow in two dimensions, cellular automation models have been proposed [2,3]. Asymmetric hopping processes have also been applied to model traffic flow on a one-dimensional road [4-6]. When the number of cars is large, traffic flows can be modelled phenomenologically in terms of a one-dimensional compressible gas [7-9]. Such an approach predicts the appearance of shock waves, where hydrodynamic quantities, such as the average density and velocity, become discontinuous. However, the hydrodynamic approach does not naturally describe the behavior of traffic flows in the low-density limit where there are large heterogeneities in traffic density. For this situation, a microscopic model may provide a more appropriate description.

In this article, we introduce a ballistic aggregation process to model the kinetics of clustering in one-dimensional traffic flows. Our approach is inspired, in part, by the recent interesting results that have been obtained for a variety of reaction processes which involve ballistic particles including: ballistic agglomeration, $A_i + A_j \to A_{i+j}$, with momentum conserving collisions [10,11]; ballistic annihilation, $A + A \to 0$, [12,13]; and several nucleation and ballistic growth processes [14-16]. In our model, cars move ballistically in a one direction, say to the right, according to an initial velocity distribution. Clusters form whenever a faster car overtakes a slower car or cluster. The overtaking car then assumes the velocity of the lead car in the cluster. This model is an idealized description for one-lane traffic flow. While there are obvious shortcomings in our model, it is exactly soluble and permits a thorough understanding of the kinetics of the aggregation process.

This paper is organized as follows. In section II, we present the model and postulate the scaling behavior for the velocity and the concentration of the clusters. This approach makes use of the statistical properties of the minimal random variable within a large sample. In section III, we investigate the distribution of cluster velocities. For this distribution, the cluster size is irrelevant and this feature allows us to consider a simpler "coalescence only" model. For this reduced problem, the velocity distribution is obtained exactly in terms of the initial distribution of car velocities and then evaluated for general continuous distributions. Building on these results, the general clustering process is solved in section IV and an asymptotically exact expression for the joint cluster mass-velocity distribution is obtained. In section V, we present a formal solution for the velocity distribution function for an inhomogeneous initial distribution of particles. We examine the temporal behavior that arises for a simple step function initial spatial distribution. In section VI, we investigate another generalization of the model to the situation with a spatially and temporally homogeneous input of cars. Depending on the functional form of the input velocity distribution in the low-velocity limit, the input can give rise to a steady state or to a system which continues to evolve indefinitely. We give our conclusions in section VII. The details of specific calculations are given in the Appendices.

II. SCALING ANALYSIS

We consider an idealized one-dimensional traffic flow in which the size of each car is zero. This is appropriate for describing clustered traffic in the low-density limit, a situation which is often encountered on rural secondary roads. In the following, we will refer to such sizeless cars as particles. We consider the initial condition when there are only isolated particles ("monomers") in the system with a random spatial distribution of density c_0. The initial velocity distribution, $P(v,t=0)$, can generally be written in the scaling form

$$P(v,t=0) = \frac{c_0}{v_0} P_0\left(\frac{v}{v_0}\right) \qquad v>0, \tag{1}$$

with $\int_0^\infty P_0(z)dz = 1$. Here we have tacitly subtracted the assumed finite value of the velocity of the slowest car from all velocities. In what follows, it is often convenient to introduce the dimensionless density $c/c_0 \to c$, velocity $v/v_0 \to v$ and time $c_0 v_0 t \to t$. This yields a rescaled initial concentration which is equal to unity.

In our model, particles move at their initial velocities and whenever a particle overtakes a cluster aggregation occurs. The aggregation rule is simply that two colliding clusters form a new cluster with a velocity equal to the smaller of the two incident cluster velocities and with a mass equal to the sum of the two cluster masses (Figure 1). If we denote a cluster of mass m and velocity v by $A_{m,v}$, the process is described by the reaction scheme

$$A_{m_1,v_1} + A_{m_2,v_2} \to A_{m_1+m_2,\min\{v_1,v_2\}}. \tag{2}$$

We present now a simple argument, based on the statistics of extremes, to predict the asymptotic time dependence of the typical cluster mass m and typical cluster velocity v at time t. Since the typical distance, l, between clusters grows with time as $l \sim vt$, the typical number of particles in a cluster is proportional to this distance, yielding $m \sim l \sim vt$. To find the typical velocity, one has to relate the mass of a cluster to its velocity. Such a relation may be found exactly for an auxiliary "one-sided" problem in which particles are placed with a fixed density c_0 to the left of a given "target" particle which moves at velocity v, and no particles are placed to the right. Eventually, this target particle will form a cluster that includes all consecutive particles to its left whose initial velocities are larger than v. The probability that there are exactly k such particles is equal to $P_- P_+^k$, with $P_+(v)$ $\big(P_-(v)\big)$ defined as the probability that a particle has a velocity larger (smaller) than v, *i. e.*, $P_+(v) = \int_v^\infty P_0(v')dv'$. Therefore, the average number of particles in the cluster that ultimately forms is given by

$$\langle m(v)\rangle = \sum_1^\infty k P_- P_+^k = P_+/P_-. \tag{3}$$

Let us now assume a power-law behavior of the initial velocity distribution for small velocities,

$$P_0(v) \simeq a v^\mu, \qquad v \ll 1 \tag{4}$$

with $\mu > -1$ for normalizability. Imposing this power-law form in Eq. (3) yields

$$\langle m(v)\rangle = \frac{P_+(v)}{P_-(v)} \propto \frac{1}{v^{\mu+1}}, \tag{5}$$

for sufficiently low velocities. For a particle which moves with the typical velocity, it is reasonable to expect that this result for the "one-sided" problem gives the correct behavior for the full "two-sided" problem. If we combine Eq. (5) with our previous estimate $m \sim vt$, we find the following asymptotic relations,

$$\begin{aligned} m &\sim t^\alpha \qquad \text{with} \qquad \alpha = \frac{\mu+1}{\mu+2} \\ v &\sim t^{-\beta} \qquad \text{with} \qquad \beta = \frac{1}{\mu+2} \end{aligned} \tag{6}$$

Since the mass is conserved in the aggregation process, the typical cluster mass and the concentration of clusters c are related by $c \sim 1/m \sim t^{-\alpha}$. Notice that in the limit $\mu \to \infty$, the mass grows linearly with time. In contrast, when $\mu \to -1$, the mass is roughly constant, since the velocity distribution becomes effectively unimodal and collisions are exceedingly rare. This qualitative dependence on the form of the initial velocity distribution is reminiscent of the ballistic annihilation process [13], where ballistically moving particles annihilate upon collision. In both processes, one finds that the fundamental exponents are related by $\alpha + \beta = 1$ as a consequence of the relation $c \sim 1/vt$. Moreover, for the two processes the decay exponents have similar functional dependences on the form of the initial velocity distribution. However, the values of the decay exponents are different: for example, for a uniform distribution (corresponding to $\mu = 0$) one finds $\alpha = 1/2$ for the traffic model, while $\alpha \cong 0.76$ is obtained in simulations of the annihilation process. Additionally, despite the qualitative similarity between these two models for continuous velocity distributions, different behaviors occur when the velocities are discrete. For such discrete distributions, the concentration typically decays algebraically in time for ballistic annihilation [13], while the concentration decays exponentially in time for the traffic model.

Since both the typical mass and the velocity scale as power laws in time, the probability of finding a cluster of mass m and velocity v, $P_m(v,t)$, is expected to evolve toward a scaling distribution. Taking into account mass conservation, $\int_0^\infty dv \sum_m mP_m(v,t) = \text{const.}$, we postulate the scaling form

$$P_m(v,t) \simeq t^{\beta-2\alpha}\Phi(M,V), \tag{7}$$

for the joint distribution, where the scaled mass, M, and scaled velocity, V, are defined by

$$M = m/t^{\alpha} \qquad \text{and} \qquad V = vt^{\beta}. \tag{8}$$

Note that while the mass m is a discrete variable, the rescaled mass M is continuous.

Once the joint mass-velocity distribution is found, the single-variable mass and velocity distributions can be found by suitable integrations over the subsidiary variable. Thus the velocity distribution, $P(v,t) = \sum_m P_m(v,t)$, should have the scaling form

$$P(v,t) \simeq t^{\beta-\alpha}\psi(V), \tag{9a}$$

with $\psi(V) = \int_0^\infty dM\,\Phi(M,V)$, while the cluster-mass distribution, $P_m(t) = \int_0^\infty P_m(v,t)$, should have the scaling form

$$P_m(t) \simeq t^{-2\alpha}\phi(M), \tag{9b}$$

with $\phi(M) = \int_0^\infty dV\,\Phi(M,V)$.

III. THE CAR SURVIVAL PROBABILITY

As a preliminary step in obtaining a full solution for traffic clustering, consider the velocity distribution $P(v,t)$. For this quantity, we can ignore the masses of each cluster and focus only on the survival probability of a given car. Thus the evolution of the velocity distribution is governed by the "derived" coalescence process

$$A_{v_1} + A_{v_2} \to A_{\min\{v_1,v_2\}}. \tag{10}$$

In the coalescence process, the density of particles with velocity v is identical to $P(v,t)$, the velocity distribution of clusters in the full traffic aggregation model defined by Eq. (2).

Let $S(v,t)$ be the survival probability of particles of velocity v at time t. Here "survival" means a car does not overtake any traffic, but an overtaken car is defined to have survived. Then the velocity distribution function is given by

$$P(v,t) = P_0(v)S(v,t). \tag{11}$$

The survival probability $S(v,t)$ can be found by considering the possible collisions of a particle with initial position and velocity (x,v) with slower particles whose initial positions are to the right of x. A collision between the initial particle with co-ordinates (x,v) and a slower v'-particle does not occur up to time t if the interval $[x, x+(v-v')t]$ does not include the slower particle. For an initial velocity distribution, $P_0(v)$, and a Poissonian initial spatial distribution, the probability that there is no particle with velocity between v' and $v'+dv'$ in the interval $[x, x+(v-v')t]$ is

$$\exp\left[-dv' P_0(v')(v-v')t\right]. \tag{12}$$

For a particle to survive to time t, this exclusion probability should be taken into account for every $v' < v$. To verify this, let us assume otherwise and derive a contradiction. Thus consider a particle with initial data (x,v) that has maintained its original velocity to time t. In addition, assume that a slower v'-particle is initially present in the above exclusion zone, *i. e.*, $\Delta x(0) < \Delta v(0)t$. Here $\Delta x(t)$ is the distance between the two particles and $\Delta v(t)$ the relative velocity at time t. Since the velocity v' can only decrease over time due to collisions, one has $\Delta v(t) \geq \Delta v(0)$. Consequently, at time t, the separation between the two particles, $\Delta x(t) = \Delta x(0) - \int_0^t \Delta v(t')dt' \leq \Delta x(0) - \Delta v(0)t < 0$. Thus the v-particle does not survive, in contradiction with the original assumption.

Hence, the survival probability is simply a product of the exponential factors of Eq. (12) for all v', with $v' < v$. Evaluating this product gives the survival probability

$$S(v,t) = \exp\left[-t\int_0^v dv'(v-v')P_0(v')\right], \tag{13}$$

and combining with Eq. (13) yields the velocity distribution,

$$P(v,t) = P_0(v)\exp\left[-t\int_0^v dv'(v-v')P_0(v')\right]. \tag{14}$$

This is valid for an *arbitrary* initial velocity distribution $P_0(v)$; the only source of stochasticity arises from the initial conditions. For discrete initial distributions it is seen from Eq. (14) that the approach to the final concentration is exponential in time. Thus, we focus only on the more interesting continuous initial velocity distributions.

For the power-law initial velocity distribution $P_0(v) \simeq av^\mu$ for $v \ll 1$, a direct calculation shows that the long-time velocity distribution approaches a form that is independent of the details of the large-velocity tail of the initial distribution,

$$P(v,t) \simeq av^\mu \exp\left[-btv^{\mu+2}\right], \tag{15}$$

with $b = a/(\mu+1)(\mu+2)$. This expression can be written in the scaling form (9a) with the scaling function

$$\psi(V) = aV^\mu \exp\left[-bV^{\mu+2}\right]. \tag{16}$$

From this solution we see that the velocity distribution maintains the original power-law form for small velocities. The exact solution also validates the scaling assumption that the asymptotic decay as well as the shape of the limiting distribution are determined solely by the low-velocity tail of the initial distribution which, in turn, is governed by the exponent μ.

From Eq. (15), it is straightforward to compute the total concentration and the average cluster velocity,

$$c(t) = \int_0^\infty dv\, P(v,t), \quad \langle v(t)\rangle = \frac{1}{c(t)}\int_0^\infty dv\, v\, P(v,t). \tag{17}$$

This gives

$$c(t) \simeq (\mu+1)b\Gamma(\alpha)(bt)^{-\alpha}, \tag{18}$$

and

$$\langle v(t)\rangle \simeq \frac{1}{\Gamma(\alpha)}(bt)^{-\beta}, \tag{19}$$

respectively. These expressions confirm the scaling laws suggested in Eq. (6).

Interestingly, the exact solution of Eq. (14) satisfies the following Boltzmann-like integro-differential equation,

$$\frac{\partial P(v,t)}{\partial t} = -P(v,t)\int_0^v dv'(v-v')P(v',0). \tag{20}$$

This equation suggests that the loss of v-particles due to collisions with slower v'-particles occurs at a rate proportional to the relative velocity, $(v-v')$. Moreover, the pair correlation function factorizes into a product of single-particle velocity distributions, $P(v,v',t) = P(v,t)P(v',0)$ but with different time arguments for the two factors. In contrast, in the conventional Boltzmann equation, the decomposition would involve the same argument for each velocity distribution. Thus, the exact Eq. (20) quantitatively indicates the degree of approximation of the mean-field Boltzmann equation.

IV. THE FULL PROBABILITY DISTRIBUTION

We now solve for the joint mass-velocity distribution function for the general traffic model. To obtain $P_m(v,t)$, the density of clusters of mass m and velocity v, it is useful to introduce the cumulative distribution, $Q_m(v,t)$, the distribution of clusters of velocity v and mass greater than or equal to m. Once the latter distribution is known, $P_m(v,t)$ can be obtained by

$$P_m(v,t) = Q_m(v,t) - Q_{m+1}(v,t). \tag{21}$$

Notice that the density of clusters of mass greater or equal to one is equal to the total cluster density, $Q_1(v,t) = P(v,t)$.

Consider a cluster of velocity v which contains at least m particles. Let us number the consecutive particles in a cluster from right to left by the index i and denote the rightmost particle as $i = 0$. Denote the initial distance between the $i^{\rm th}$ and $(i-1)^{\rm th}$ particle as x_i, as illustrated in Fig. 2. We first solve for $Q_{m=2}(v,t)$ and then generalize to any m. Since $Q_2(v,t)$ is the probability that a cluster of velocity v has at least two particles at time t, it is equal to the product of the probability that the particle $i = 0$ has survived up to time t, $P(v,t)$, and the probability that the cluster $i = 1$ (whose mass may be larger than unity) collides with the particle $i = 0$ prior to time t.

For this collision to occur, the collision partner from the left ($i = 1$) must have a velocity larger than v and the interval $x_1 < (v_1 - v)t$ must be free of other clusters. The probability for this composite event is simply the product of each individual event. Since an interval of length x_1 is empty with probability $\exp(-x_1)$, the collision probability is

$$Q_2(v,t) = P(v,t) \int_v^\infty dv_1 P_0(v_1) \int\limits_{x_1<(v_1-v)t} dx_1 \exp(-x_1). \tag{22}$$

The fact that the v_1-particle cannot be slowed down by any other particle before colliding with the v-particle is crucial in obtaining the solution.

To derive $Q_m(v,t)$ for general m, the joint velocity distance distribution $P_0(v_i)\exp(-x_i)$ is integrated over the position and velocity of the $i^{\rm th}$ particle for $i = 1, \ldots, m-1$. To ensure a collision, all $m-1$ particles have to move faster than the lead particle and the distance of the $i^{\rm th}$ particle from the lead particle must obey $x_1 + \cdots + x_i \le (v_i - v)t$. Imposing these constraints on the integration over the velocity and distance of the $n-1$ trailing particles yields the formal exact expression for the cumulative mass-velocity distribution,

$$Q_m(v,t) = P(v,t) \prod_{i=1}^{m-1} \int_v^\infty dv_i \int\limits_{x_1+\cdots+x_i<(v_i-v)t} dx_i P_0(v_i)\exp(-x_i). \tag{23}$$

For the initial velocity distribution given by Eq. (4), we find the following asymptotic behavior (see Appendix A)

$$Q_m(v,t) \simeq t^{\beta-\alpha} a V^\mu \exp\left[-b(V+M)^{\mu+2}\right], \tag{24}$$

in terms of the scaling variables $M = m/t^\alpha$ and $V = vt^\beta$.

In the long time limit, the joint mass-velocity distribution, $P_m(v,t) = Q_m(v,t) - Q_{m+1}(v,t)$ can be approximated by $P_m \approx -\partial Q_m/\partial m$. Performing the differentiation gives,

$$P_m(v,t) \simeq t^{\beta-2\alpha} a\, b\, (\mu+2) V^\mu (V+M)^{\mu+1} \exp\left[-b(V+M)^{\mu+2}\right], \tag{25}$$

which has been explicitly written in the asymptotic scaling form of Eq. (7). This result provides a complete description of the traffic aggregation process. It may be considered as the ballistic counterpart of the well-known result [17] for diffusion-controlled aggregation in one dimension.

For arbitrary μ we are unable to evaluate the integral over the velocity and obtain the explicit mass distribution. However, for the particular case of a uniform initial velocity distribution, $\mu = 0$, it is straightforward to show that

$$\phi(M) = a \exp(-aM^2/2) \qquad \mu = 0. \tag{26}$$

Another amusing feature of the joint distribution function $\Phi(M,V)$ for $\mu = 0$ is the symmetry with respect to the variables V and M. Thus the cluster mass distribution, $\phi(M)$, and the cluster velocity distribution, $\psi(V)$, are identical Gaussian functions.

Generally, we are able to extract only asymptotic behavior from Eq. (23). However, in the special case of exponential distribution, $P_0(v) = e^{-v}$, one can perform all the integrations and obtain an explicit solution as detailed in Appendix B.

V. CLUSTERING IN HETEROGENEOUS TRAFFIC FLOW

The above approach can be generalized to the case of a spatially heterogeneous initial velocity distribution, $P_0(x,v)$. For simplicity, we ignore the masses of clusters and limit ourselves to studying the velocity distribution. This time and space dependent velocity distribution, $P(x,v,t)$, may be found by a straightforward generalization of the approach developed in section III for the spatially homogeneous case. The resulting expression for $P(x,v,t)$ reads

$$P(x,v,t) = P_0(x-vt,v)\exp\left[-\int_0^v dv' \int_{x-vt}^{x-v't} dx' P_0(x',v')\right]. \tag{27}$$

As an illustrative example of the effects of a heterogeneous initial particle distribution, consider the one-sided distribution in which particles are placed with a fixed density to the left of the origin and there are no particles to the right. Thus $P_0(x,v) = \theta(-x)P_0(v)$, with θ the Heaviside step function. For this initial distribution, Eq. (27) yields

$$P(x,v,t) = \theta(vt-x)P_0(v)\exp\left[-t\int_{x/t}^v dv'(v-v')P_0(v') - (vt-x)\int_0^{x/t} dv' P_0(v')\right]. \tag{28}$$

In the long-time limit, the average velocity decays as $t^{-\beta}$ and hence the front propagates as $x = vt \sim t^\alpha$. Since the velocity and the position of the front scale as power laws in time, Eq. (28) can be expected to have a scaling form. Indeed, by introducing the scaled variables $X = x/t^\alpha$ and $V = vt^\beta$, one can recast Eq. (28) into the scaling form

$$P(x,v,t) \simeq t^{\beta-\alpha}\Psi(X,V), \tag{29}$$

with the scaling function

$$\Psi(X,V) = \theta(V-X)aV^\mu \exp\left[-b(V^{\mu+2} - X^{\mu+2})\right]. \tag{30}$$

When $X = 0$, this scaling function coincides with Eq. (16), the velocity distribution for the homogeneous case, $\Psi(0,V) = \psi(V)$. Notice also that the density of clusters at scaled position X, $c(X,t)$, equals

$$c(X,t) = c(t)\int_X^\infty dV\,\Psi(X,V) \Big/ \int_0^\infty dV\,\psi(V), \tag{31}$$

with $c(t)$ given by Eq. (18). In the large X limit, Eqs. (30) and (31) yield

$$c(X,t) \simeq \frac{\mu+1}{X}t^{-\alpha}, \quad \text{for} \quad X \gg 1. \tag{32}$$

Consider now the total number of clusters that infiltrate the initially empty positive half-line, $N(t) \equiv t^\alpha \int_0^\infty c(X,t)\,dX$. The asymptotic behavior of this quantity is actually determined by the finite upper cutoff of the integral, which in turn is given by the position of the rightmost particle. For such particles the velocity is of order unity and hence $X_{\rm upper} = x_{\rm max}/t^\alpha \sim vt/t^\alpha \sim t^\beta$. Therefore

$$N(t) \simeq \int^{t^\beta} \frac{\mu+1}{X} dX \simeq \alpha\log(t). \tag{33}$$

Thus the number of clusters entering the empty half-line grows only logarithmically with time for arbitrary initial velocity distributions. The only dependence on $P_0(v)$ in Eq. (33) is the prefactor $\alpha = (\mu+1)/(\mu+2)$.

VI. CLUSTERING IN TRAFFIC FLOW WITH INPUT

In this section, we investigate traffic clustering when there is a spatially uniform input of cars. This generalization is motivated by real traffic where cars may enter and exit a roadway. For the specific case of a spatially homogeneous input of cars we can determine the velocity distribution using techniques similar to those employed for the traffic coalescence model with no input.

Denote by $R(v,t)$ the input rate of particles with velocity v at time t per unit length. The velocity distribution function for this system, $P(v,t)$, can be expressed as a convolution of the flux and the the probability that a particle which was injected at time t' maintains its velocity v up to time t in the presence of the input, $S_I(v,t,t')$,

$$P(v,t) = \int_0^t dt'\,R(v,t')\,S_I(v,t,t'). \tag{34}$$

In writing this expression, we have assumed that the system is initially empty. A particle which was injected at time t' will survive until time t if it avoids collisions with all slower particles which were present in the system at time t', as well as avoids collisions with all particles which are injected at later times $t'' > t'$. The probability for this composite event is simply the product of the probabilities of each event,

$$S_I(v,t,t') = \exp\left[-\int_0^v dv'(v-v')\left\{P(v',t')(t-t') + \int_{t'}^{t} dt'' R(v',t'')(t-t'')\right\}\right]. \tag{35}$$

Here the first factor, obtained from Eq. (13) by replacing $P_0(v')$ with $P(v',t')$, yields the probability of avoiding collisions with particles injected prior to time t'. The second factor represents the product of exclusions of the type off Eq. (12) for times larger than t' and velocities smaller than v'. This factor accounts for the probability that there are no collisions with particles injected at times $t'', t'' > t'$. Note that the kernel of the second factor involves the input rate R at time t'', which plays the role of the initial velocity distribution at this instant of time. Substituting Eq. (34) into Eq. (35) gives a nonlinear integral equation that describes the kinetics of traffic clustering in the presence of homogeneous particle input. In the following, we will assume that the flux is constant *both* in time and space, $R(v,t) = P_0(v)$ with $\int_0^\infty dv P_0(v) = 1$, so that the governing integral equation is

$$P(v,t) = P_0(v)\int_0^t dt' \exp\left[-\frac{1}{2}t'^2\int_0^v dv'(v-v')P_0(v) - t'\int_0^v dv'(v-v')P(v',t-t')\right]. \tag{36}$$

In parallel with the case of no input, consider again an initial velocity distribution with a power-law small-velocity tail, $P_0(v) \sim v^\mu$. Furthermore, let us assume that the concentration and velocity continue to vary as $c \sim t^{-\alpha}$ and $v \sim t^{-\beta}$, respectively, and that the velocity distribution continues to have the scaling form of Eq. (9), $P(v,t) \sim t^{\beta-\alpha}\psi(V)$. Substituting these into Eq. (36), one can extract consistency conditions for the exponents α and β. For example, the powers of time on both sides of Eq. (36) should be equal – hence $\beta - \alpha = -\beta\mu + 1$. Furthermore, both terms in the exponential in the right-hand side of Eq. (36) cannot depend on time explicitly – hence $\beta(\mu+2) = 2$ and $\alpha + \beta = 1$, respectively. These conditions yield $\alpha = \mu/(\mu+2)$ and $\beta = 2/(\mu+2)$. Notice, however, that when μ is positive the exponent α is also positive and therefore the concentration, $c \sim t^{-\alpha}$, decays to zero. This is in obvious contradiction with the nature of the problem: with a constant flux the concentration may grow indefinitely or a steady state concentration may be reached. Thus, one can expect that the above description holds only for $\mu < 0$. For positive μ, we anticipate that the system reaches a steady state with a constant concentration and a typical cluster mass $n \sim t$. At the transition $\mu = 0$, a logarithmic temporal dependence is anticipated to occur.

While we cannot confirm the above picture rigorously, we can provide heuristic justification. First *assume* that the velocity distribution evolves towards a steady state $P_\infty(v)$. Then as $t \to \infty$ Eq. (36) becomes

$$P_\infty(v) = P_0(v)\int_0^\infty dt' \exp\left[-\frac{1}{2}t'^2\int_0^v dv'(v-v')P_0(v) - t'\int_0^v dv'(v-v')P_\infty(v')\right]. \tag{37}$$

If $P_0(v) \sim v^\mu$ as $v \to 0$, Eq. (37) suggests a similar behavior for the steady state velocity distribution function,

$$P_\infty(v) \simeq Av^\nu \quad v \ll 1. \tag{38}$$

Since the concentration of clusters tends to the steady state limit $c_\infty = \int_0^\infty P_\infty(v)\,dv$, the exponent ν must satisfy the inequality $\nu > -1$. If one substitutes the assumed power law behaviors for $P_0(v)$ and $P_\infty(v)$ into Eq. (37), three possibility arise in the limit $v \to 0$ which depend on the sign of $\nu - \frac{\mu}{2} + 1$. In the case where $\nu > \frac{\mu}{2} - 1$, the first exponential factor in Eq. (37) provides the dominant contribution. However, a simple calculation of the integral shows that $\nu = \frac{\mu}{2} - 1$. Similarly, for $\nu < \frac{\mu}{2} - 1$, one again finds $\nu = \frac{\mu}{2} - 1$. Only the last possibility, $\nu = \frac{\mu}{2} - 1$, appears to be self-consistent. Since $\nu > -1$, we obtain $\mu > 0$. Therefore starting from the assumption that the system reaches the steady state we have obtained that the exponent μ should be positive. This provides evidence for our conclusion that $\mu = 0$ demarcates the scaling and steady state behaviors.

Notice also that for $0 < \mu \ll 1$, an asymptotic analysis of Eq. (37) gives the numerical prefactor in Eq. (38), $A \simeq \sqrt{a\mu/2}$. This yields the estimate for the steady state concentration of clusters,

$$c_\infty \simeq \sqrt{\frac{2a}{\mu}} \quad 0 < \mu \ll 1, \tag{39}$$

Since c_∞ diverges as $\mu \to 0$ this indicates that at the critical value $\mu, \mu = 0$, the system is still evolving.

Assuming the scaling form, obtained by the power counting analysis of Eq. (36), let us examine the asymptotic behavior of the velocity distribution and the typical concentration. If we substitute the scaling assumptions

$$P(v,t) \simeq t^{\beta-\alpha}\psi(V) \quad \text{with} \quad \alpha = \frac{\mu}{\mu+2}, \quad \beta = \frac{2}{\mu+2}, \tag{40}$$

into Eq. (36), we arrive at the following equation for the scaling function $\psi(V)$,

$$\psi(V) = aV^{\mu} \int_0^1 d\tau \exp\left[-\frac{1}{2}b\tau^2 V^{\mu+2} - \tau(1-\tau)^{\beta-\alpha} \int_0^V \psi[V'(1-\tau)^{\beta}](V-V')dV'\right], \tag{41}$$

where $\tau = t'/t$ and $b = a/(\mu+1)(\mu+2)$.

Although we are unable to solve this nonlinear integral equation in general, we can obtain interesting information regarding the interesting borderline case of $\mu = 0$. This case corresponds to $\alpha = 0$, suggesting that the concentration grows slower than algebraically in time. However, the concentration of clusters is given by

$$c(t) \simeq t^{-\alpha} \int \psi(V)dV, \tag{42}$$

and for $\mu = 0$ the integral diverges at the upper limit. To obtain the asymptotic behavior we consider Eq. (41) for the case $\mu = 0$,

$$\psi(V) = a \int_0^1 d\tau \exp\left[-\frac{1}{2}b\tau^2 V^2 - \tau(1-\tau) \int_0^V \psi[V'(1-\tau)](V-V')dV'\right]. \tag{43}$$

If we temporarily ignore the second term in the exponent, we find $\psi(V) \sim V^{-1}$ for $V \to \infty$. If we then include the second term in the exponent and apply the previous asymptotic behavior of $\psi(V) \sim V^{-1}$, we find $\psi(V) \sim (V \log V)^{-1}$. These estimates suggest the ansatz

$$\psi(V) \simeq CV^{-1}(\log V)^{-\lambda}, \tag{44}$$

for $V \gg 1$. Upon substituting Eq. (44) into Eq. (43) one obtains the constants $\lambda = 1/2$ and $C = 1/\sqrt{2}$. With these values, the integral in the right-hand side of Eq. (42) diverges as $\sqrt{2\log V}$, where V now denotes the maximum value of the scaled velocity. Since $\beta = 1$, this maximal velocity is proportional to t, which therefore suggests the logarithmic time dependence

$$c(t) \simeq \sqrt{2\log t}. \tag{45}$$

In the complementary case of $\mu < 0$, we have confirmed that the naive scaling ansatz is consistent with Eq. (42).

VII. CONCLUSION

We have introduced a simple ballistic aggregation process that mimics the kinetics of clustering in a single lane of traffic. Through direct probabilistic approaches, the analytical forms of the cluster velocity distribution and the joint mass-velocity distribution have been derived. For an initial velocity distribution of the form $P_0(v) \sim v^{\mu}$ as $v \to 0$, both the average velocity and the average cluster mass have power law time dependences with exponents that are rational functions of μ. This qualitative behavior is similar to that observed in the closely related ballistic annihilation process. We are also able to determine the asymptotic form of the joint mass-velocity distribution.

Our model can also be analyzed in the cases of a spatially heterogeneous particle distribution and continuous input of particles. For the simple case of an initial one-sided spatial distribution, the system evolves towards a scaling distribution both in velocity and spatial variables. We have thus found that the total number of clusters in the initially empty half-line grows logarithmically with time for all initial velocity distributions. When there is a steady input of particle in an initially empty system, we have found that there is a transition between steady state behavior for $\mu > 0$ and transient behavior for $\mu < 0$ which is similar to that found when there is no in put. For the borderline case of the uniform distribution, $\mu = 0$, we have found that the total concentration of clusters grows as $\sqrt{\log t}$.

The irreversible traffic model introduced in this paper leads to ever-growing clusters. To describe traffic flows more realistically, several mechanisms to induce a steady state can be envisioned. For example, the input model can be generalized to incorporate a flux out of the system. Another realistic direction is to allow a faster car to pass a slower car at a rate which is some increasing function of the velocity difference of the two cars. This would allow a fast car to traverse a cluster car-by-car and ultimately regain its intrinsic velocity once the cluster is completely passed. It may prove interesting to examine the steady-state properties for this class of models.

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APPENDIX A: DERIVATION OF EQ. (24)

In this appendix, we derive the asymptotic form of the cumulative velocity-mass distribution which is valid for an arbitrary initial velocity distribution with a power-law small velocity tail. The starting point for calculation of $Q_m(v,t)$ is the formal expression of Eq. (23). By interchanging the order of the velocity and spatial integrations, the expression can be rewritten as

$$Q_m(v,t) = P(v,t) \prod_{i=1}^{m-1} \int_0^\infty dx_i \exp(-x_i) \int_{v+(x_1+\cdots+x_i)/t}^\infty dv_i \, P_0(v_i). \tag{A1}$$

Since the details of the initial distribution for large velocities do not change the form of the scaling solution, we may treat the more general power-law case by choosing a specific initial distribution of velocities whose form is convenient for performing the integration over the velocity in the right-hand side of Eq. (A1). Since the velocity distribution near $v = 0$ has a power-law tail, the most convenient initial distribution is

$$P_0(v) = av^\mu \exp\left(-av^{\mu+1}/(\mu+1)\right). \tag{A2}$$

For this initial distribution, the integration over the velocity variables is immediate and one finds,

$$Q_m(v,t) = P(v,t) \prod_{i=1}^{m-1} \int_0^\infty dx_i \exp(-x_i) \exp\left[-a\Big(v + (x_1+\cdots+x_i)/t\Big)^{\mu+1}/(\mu+1)\right]. \tag{A3}$$

To evaluate the multiple spatial integral, we define $f(z) \equiv \exp\left(-az^{\mu+1}/(\mu+1)\right)$ and expand $f(z)$ to first order about the point $z = v$ and then exploit a number of simplifications associated with performing the integrals over the factors e^{-x_i}. This gives

$$\begin{aligned} Q_m(v,t) &= \prod_{i=1}^{m-1} \int_0^\infty dx_i \, e^{-x_i} f\big(v + \epsilon(x_1+\cdots+x_i)\big) \\ &\approx \prod_{i=1}^{m-1} \int_0^\infty dx_i \, e^{-x_i} f(v) + \big(\epsilon(x_1+\cdots+x_i)\big) f'(v), \\ &\approx \prod_{i=1}^{m-1} f(v+\epsilon i) + O(\epsilon^2), \end{aligned} \tag{A4}$$

where $\epsilon = 1/t$. By substituting the explicit functional form, $f(z) = \exp\big(-az^{\mu+1}/(\mu+1)\big)$, we have

$$Q_m(v,t) = P(v,t) \prod_{i=1}^{m-1} \exp\left(-a(v+i/t)^{\mu+1}/(\mu+1)\right). \tag{A5}$$

Finally, the product of the exponential factors is written as an exponent of a sum. In the limit of large m, this sum is equivalent to the integral $a\int_0^{m-1} dy (v+y/t)^{\mu+1}/(\mu+1)$. Evaluating this integral, the asymptotic form of the cumulant mass-velocity density is obtained as

$$Q_m(v,t) \simeq av^\mu \exp\left[-b(vt^\beta + mt^{-\alpha})^{\mu+2}\right]. \tag{A6}$$

In evaluating this asymptotic expression, the exponential factor of $P(v,t)$ cancels the factor that emerges from lower limit of the integration over y. In Eq. (24), the above expression is written as a scaling function.

APPENDIX B: ANALYTICAL SOLUTION FOR THE EXPONENTIAL INITIAL VELOCITY DISTRIBUTION

We outline here the explicit analytical solution for $Q_m(v,t)$ for the exponential initial velocity distribution, $P_0(v) = e^{-v}$, which corresponds to the special case $\mu = 0$ and $a = 1$. For this exponential distribution, the integration of $P_0(v_i)$ over the velocity variables v_i, according to Eq. (A1), equals $\exp[-(v + (x_1 + \cdots + x_i)/t)]$. Hence, we obtain

$$Q_m(v,t) = P(v,t)e^{-(m-1)v} \prod_{i=1}^{m-1} \int_0^\infty dx_i \exp\left[-x_i\left(1 + \frac{m-i}{t}\right)\right]. \tag{B1}$$

Upon integration over the space variables, the following exact expression is found for $Q_m(v,t)$,

$$Q_m(v,t) = P(v,t)e^{-(m-1)v}t^{m-1}\frac{\Gamma(t+1)}{\Gamma(t+m)}, \tag{B2}$$

where $\Gamma(z) = \int_0^\infty x^{z-1}\exp(-x)$ is the Euler gamma function and the velocity distribution obtained from Eq. (14) is $P(v,t) = \exp\big[-v - t\big(e^{-v} - 1 + v\big)\big]$.

The exact forms for the joint mass-velocity distribution and for the mass distribution can be evaluated from Eq. (B2) by taking the appropriate limits $m \to \infty$ and $v \to 0$. The resulting asymptotic expressions are identical with the expressions of Eqs. (25) and (26) respectively.

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FIGURE CAPTIONS

Fig. 1 Schematic illustration of the irreversible traffic model. A faster car overtakes a slower car and after the encounter, the faster car assumes the velocity of the slower car.

Fig. 2 Illustration of the initial configuration of a possible three car cluster.