Kinetic Theory of Traffic Flows

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Abstract. We describe traffic flows in one lane roadways using kinetic theory, with special emphasis on the role of quenched randomness in the velocity distributions. When passing is forbidden, growing clusters are formed behind slow cars and the cluster velocity distribution is governed by an exact Boltzmann equation which is linear and has an infinite memory. The distributions of the cluster size and the cluster velocity exhibit scaling behaviors, with exponents dominated solely by extremal characteristics of the intrinsic velocity distribution. When passing is allowed, the system approaches a steady state, whose nature is determined by a single dimensionless number, the ratio of the passing time to the collision time, the two time scales in the problem. The flow exhibits two regimes, a laminar flow regime, and a congested regime where large slow clusters dominate the flow. A phase transition separates these two regimes when only the next-to-leading car can pass.

1 Introduction

Traffic flows are strongly interacting many-body systems. Therefore, theoretical techniques such as kinetic theory and hydrodynamics are useful in describing the rich phenomenology of traffic flows which includes shock waves, phase transitions, clustering, metastability, hysteresis, etc. Traffic is typically modeled within macroscopic descriptions such as hydrodynamics and kinetic theory [1–5] or microscopic approaches, e.g., cellular automata [6–12] and car-following models [13–17]. The large body of recent work on the physics of traffic flows is surveyed in Refs. [18–20].

In this review, we describe how quenched randomness in the car velocities leads to formation of clusters in one lane roadways. We assume ballistic motion with infinite memory, namely that each car has a preferred "intrinsic" velocity by which it drives in the absence of other cars [21–26]. While the emerging behavior is quite similar to that found in stochastic particle hopping processes with quenched disorder [27–33], these ballistic motion models often admit deeper analytical treatment. Our starting point is an idealized no passing flow, where an exact analytical solution is possible, and an exact kinetic theory can be constructed [22]. We then treat more realistic generalizations where passing is allowed using approximate kinetic theories [23–26]. Our goal is to provide a concise summary where key features are emphasized and outstanding open issues are highlighted.

2 No passing zones

Our basic traffic model mimics cluster formation (often also called platoon formation) in one-lane roadways where passing is forbidden [21,22]. In this model, each car moves ballistically at its initial velocity until it overtakes the preceding car or cluster. After this encounter, the incident car assumes the velocity of the cluster which it has just joined. Cars are taken to be size-less, and collisions to be instantaneous. We primarily consider spatially homogeneous situations where the positions and the velocities of the cars are initially uncorrelated. Specifically, cars are distributed randomly in space with a concentration c_0, and their velocities are drawn from the initial velocity distribution $P_0(v)$. Remarkably, analytic expressions can be obtained for the velocity distribution and the joint size-velocity distribution of clusters. Furthermore, it is also possible to describe analytically spatial inhomogeneities and even input of cars into the system.

2.1 The velocity distribution

We first consider the cluster velocity distribution. In this description, only the lead car is relevant and trailing cars in a cluster can be ignored. Let $P(v,t)$ be the distribution of clusters with velocity v at time t. Initially, all cars are lead cars and the cluster (or lead-car) velocity distribution equals $P(v,t) = P_0(v)S(v,t)$, a product of the initial velocity distribution and the survival probability $S(v,t)$. The survival probability is the probability that a car with initial velocity v avoids "collisions" with slower cars up to time t, and hence, is still moving at the same velocity. Consider a car of initial velocity v. To ensure that it would not overtake slower cars of velocity $v' < v$, an interval of size $(v - v')t$ ahead of it must not contain v'-cars initially. For the velocity distribution, $P_0(v)$, and a Poissonian initial spatial distribution, the probability for such an event is $\exp\left[-t\,(v - v')P_0(v')\right]$. For a car to survive to time t, this exclusion probability should be taken into account for every $v' < v$, and taking the product over all possible slower cars yields the survival probability, $S(v,t) = \exp[-t\int_0^v dv'(v - v')P_0(v')]$. Consequently, the cluster velocity distribution is found for arbitrary initial conditions

$$P(v,t) = P_0(v)\exp\left[-t\int_0^v dv'(v - v')P_0(v')\right]. \tag{1}$$

The process is deterministic (the initial condition is the only source of randomness) and given the initial positions and velocities of the cars, the state of the system at any later time follows. This is reflected nicely in Eq. (1).

The exact solution (1) satisfies the following Boltzmann equation,

$$\frac{\partial P(v,t)}{\partial t} = -P(v,t)\int_0^v dv'(v - v')P(v',0). \tag{2}$$

Interestingly, this rate equation is *linear* in the velocity distribution $P(v,t)$. The collision rate is proportional to the relative velocity, $v-v'$, and the *initial* velocity distribution of the slower cars, a signature of the infinite memory in the system. This is a unique case where the hierarchy of evolution equations corresponding to the velocity distributions terminates at the first order.

Whereas the steady state of one-lane traffic with no passing is trivial, viz. all cars will eventually join a cluster led by the slowest car in the system, the time dependent behavior is interesting. We concentrate on the long time behavior, which is largely independent of the initial distribution of fast cars, as follows from Eq. (1). For discrete velocity distributions, the time dependence is exponential, and we focus on continuous distributions. In this case, we find directly from Eq. (1) that both the cluster concentration, $c(t) = \int dv P(v,t)$, and the average velocity, $\langle v(t)\rangle = c^{-1}\int dv\, v\, P(v,t)$, decay algebraically with time

$$\begin{aligned} c(t) &\sim t^{-\alpha} \qquad & \alpha &= \frac{\mu+1}{\mu+2}, \\ \langle v(t)\rangle &\sim t^{-\beta} \qquad & \beta &= \frac{1}{\mu+2}. \end{aligned} \tag{3}$$

The scaling exponents α and β depend only on the small-v extremal statistics [34] of the initial velocity distribution,

$$P_0(v) \simeq a v^{\mu} \qquad v \to 0, \tag{4}$$

via the cutoff exponent $\mu > -1$. The two scaling exponents are related by $\alpha + \beta = 1$ as dictated by an elementary mean free path argument: $cvt \sim 1$.

Since the number of cars is conserved, the average cluster size is inversely proportional to the concentration, $\langle m\rangle \sim 1/c$, and the size growth law is $\langle m\rangle \sim t^{\alpha}$. In the limit $\mu \to \infty$, the size grows linearly with time. In contrast, when $\mu \to -1$, the size remains roughly constant, since the velocity distribution becomes effectively unimodal and collisions become exceedingly rare. This qualitative dependence on the form of the initial velocity distribution is reminiscent of ballistic annihilation processes, where ballistically moving particles annihilate upon collision [35–38]. The above clustering process can be viewed as a ballistic aggregation process that possesses a single mass conservation law. The sensitive dependence on the initial conditions is in contrast with momentum conserving ballistic agglomeration processes (that mimics large scale formation of matter in the universe) where a universal scaling asymptotic behavior emerges [39–41].

The average velocity is the only relevant velocity scale in the problem and asymptotically the velocity distribution follows the scaling form

$$P(v,t) \simeq t^{\beta-\alpha}\Phi(vt^{-\beta}). \tag{5}$$

From Eqs. (1) and (4), the scaled distribution is $\Phi(z) = az^{\mu}\exp(-bz^{\mu+2})$, with $b = a/[(\mu+1)(\mu+2)]$. Therefore, the small-v asymptotics of the initial velocity distributions governs not only the scaling exponents but also the entire shape (including the large velocity tail) of the scaling function $\Phi(z)$. This scaling behavior indicates that at time t, most cars moving initially with velocities larger than the typical velocity scale, $\langle v(t)\rangle \sim t^{-\beta}$, have already joined clusters led by slower cars, while cars slower than this velocity scale are still driving with their initial velocity.

2.2 The size-velocity distribution

Given the nature of the model, a car is only affected by the initial configuration of cars *ahead* of it. This key feature enables solution of the cluster velocity distribution and it allows treatment of a more detailed quantity, the joint size-velocity distribution. To obtain $P_m(v,t)$, the density of clusters of size m and velocity v, it is useful to introduce the cumulative distribution, $Q_m(v,t)$, the distribution of clusters of velocity v containing at least m cars. Knowledge of this cumulative distribution yields the joint size-velocity distribution via differencing, $P_m(v,t) = Q_m(v,t) - Q_{m+1}(v,t)$.

Consider the first nontrivial quantity, $Q_2(v,t)$, the probability distribution of clusters of velocity v with at least two cars at time t. This quantity is equal to the product of the probability that lead car has survived up to time t, $P(v,t)$, and the probability that the car trailing it actually experiences a collision prior to time t. Let x_1 and v_1 be the initial position and the initial velocity of the trailing car, respectively. For such a collision to occur, the trailing car must be faster than the lead car, $v_1 > v$, and the interval separating the two cars must be initially free of other cars. The probability for this composite event is the product of the probabilities of each individual event. Given a random (Poisson) spatial distribution, the probability an interval is empty is exponential in its length, and the collision probability is

$$Q_2(v,t) = P(v,t)\int_v^\infty dv_1 P_0(v_1)\int_{x_1<(v_1-v)t} dx_1 \exp(-x_1). \tag{6}$$

The fact that the trailing car cannot be slowed down by any other car before colliding with the lead car is crucial in obtaining this solution. This solution can be generalized to arbitrary cluster sizes. Following the two-car cluster case, one simply integrates over all the initial positions and velocities of the consecutive cars to eventually collide with the lead car in the cluster to give

$$Q_m(v,t) = P(v,t)\prod_{i=1}^{m-1}\int_v^\infty dv_i\, P_0(v_i) \int\limits_{x_1+\cdots+x_i<(v_i-v)t} dx_i \exp(-x_i). \tag{7}$$

The integration limits reflect the fact that all the colliding cars must be faster than the lead car, and the restriction on the integration limits ensures that cars are sufficiently close to the lead car so that collisions indeed occur.

Given the cumulative car distribution, the joint size-velocity distribution can be formally obtained. Since $\langle m\rangle \sim t^{\alpha}$ and $\langle v\rangle \sim t^{-\beta}$, we anticipate the following scaling behavior: $P_m(v,t) \simeq t^{\beta-2\alpha}\Psi(mt^{-\alpha}, vt^{\beta})$. This indeed holds and from Eq. (7) one obtains the scaled joint distribution

$$\Psi(x,z) = cz^{\mu}(x+z)^{\mu+1}\exp\left[-b(x+z)^{\mu+2}\right], \tag{8}$$

with $c = a^2/(\mu+1)$. Again only two parameters, a and μ characterizing the small velocity characteristics of the initial conditions, are needed to fully describe the asymptotic state of the system. The joint distribution (8) provides a comprehensive description of the traffic clustering process. It may be considered as the counterpart of the well-known result for diffusion-controlled aggregation in one-dimension [42].

Integration of the scaling function with respect to x reproduces the scaled velocity distribution $\Phi(z)$. The complementary scaled size distribution cannot be found in a closed elementary form, except for the special case of asymptotically flat distributions ($\mu = 0$) where both of the single variable scaling functions are purely Gaussian.

2.3 Generalizations

A natural generalization is to spatially heterogeneous initial velocity distributions, $P_0(x,v)$. The time and space dependent cluster velocity distribution, $P(x,v,t)$, follows from a straightforward generalization of the basic derivation in the homogeneous case

$$P(x,v,t) = P_0(x-vt,v)\exp\left[-\int_0^v dv'\int_{x-vt}^{x-v't} dx' P_0(x',v')\right]. \tag{9}$$

For instance, consider the special case where cars are uniformly distributed in the region $x \le 0$ while the region ahead is empty. Here, one finds a governing length scale $x \sim vt \sim t^{\alpha}$ with the same exponent α as in Eq. (3). This length scale characterizes a propagating front of clusters, and the space dependent concentration $c(x,t)$ becomes a function of the scaling variable $X = xt^{-\alpha}$, namely $c(x,t) = t^{-\alpha}\mathcal{C}(X)$. Far from the origin, the scaled density decays as $\mathcal{C}(X) \sim X^{-1}$ implying $c(x,t) \sim x^{-1}$ for $x \gg t^{\alpha}$. Consequently, the total number of clusters in the originally empty region, $N(t) = \int_0^{\infty} dx\, c(x,t)$, grows logarithmically slow with time

$$N(t) \sim \ln t. \tag{10}$$

This growth law is universal as the dependence on the details of the initial velocity distribution is secondary, entering only via the prefactor.

In summary, a scaling asymptotic behavior characterizes the kinetics of clustering in no-passing zones of one lane roadways. The corresponding scaling exponents and scaling functions are characterized by the small-velocity statistics of the initial velocity distributions. Remarkably, it is possible to derive the exact Boltzmann equation in this case.

3 Passing zones

We now describe the complementary case of passing zones where fast cars can pass slow cars. The model we consider is a straightforward generalization of the no-passing case. The initial conditions are identical: cars are distributed randomly in space with concentration c_0 and their velocity is drawn from the intrinsic velocity distribution $P_0(v)$. The characteristic velocity scale is taken to be v_0. In the absence of other cars, cars drive ballistically with their intrinsic velocity. In the presence of other cars, two competing mechanism may cause a change in the car velocity. Collisions lead to slowing down: when a cluster overtakes a slower cluster, a larger cluster moving with the smaller of the two velocities forms. Passing leads to a velocity increase: every car inside a cluster may spontaneously pass the lead car and resume driving with its intrinsic velocity. The corresponding passing rate equals a constant, t_0^{-1}. This is a significant simplification; in realistic situations only the first few trailing cars may be able to pass.

It proves convenient to introduce dimensionless velocity, space, and time variables: $v/v_0 \to v$, $xc_0 \to x$, $c_0 v_0 t \to t$. This rescales the passing rate, $t_0^{-1} \to R^{-1}$, where

$$R = \frac{t_{\rm pas}}{t_{\rm col}} = c_0 v_0 t_0 \tag{11}$$

is the ratio of the passing time $t_{\rm pas} = t_0$ to the collision time $t_{\rm col} = (c_0 v_0)^{-1}$. We term this fundamental dimensionless quantity the "collision number" and denote it R as it is reminiscent of the Reynolds number – the small R limit is straightforward and the large R limit is characterized by boundary layers.

The starting point for kinetic theory is again the cluster velocity distribution $P(v,t)$. The approximate Boltzmann equation reads

$$\frac{\partial P(v,t)}{\partial t} = R^{-1}\left[P_0(v) - P(v,t)\right] - P(v,t)\int_0^v dv'(v-v')P(v',t). \tag{12}$$

This evolution equation assumes molecular chaos, namely that the stochastic passing events effectively mix the velocities, and therefore, spatial correlations can be neglected. This is clearly an approximation, as the collision integral does not coincide with the exact collision integral (2) derived in the no-passing case (the $R \to \infty$ limit). Still, this term reflects the fact that collisions occur only with slower clusters and that the collision rate is proportional to the velocity difference. The passing term is exact since the concentration of slowed down cars with intrinsic velocity v equals $P_0(v) - P(v,t)$.

In contrast with the no-passing case, the process is now stochastic in nature and the system approaches a nontrivial steady state. Setting the time derivative in Eq. (12) to zero we see that the steady state cluster velocity distribution $P(v) \equiv P(v, t=\infty)$ satisfies the integral equation

$$P(v)\left[1 + R\int_0^v dv'(v-v')P(v')\right] = P_0(v). \tag{13}$$

Given the intrinsic velocity distribution this relation gives the final cluster velocity distribution only implicitly. In contrast, the inverse problem is simpler as knowledge of the final distribution, the observed quantity in real traffic flows, gives explicitly the intrinsic distribution. We confirm that in the limit $R \to \infty$, all clusters move with the minimal velocity, while in the limit $R \to 0$, all cars move with their intrinsic velocity $P(v) \to P_0(v)$.

The integral equation (13) can be transformed into a differential one using the auxiliary function $Q(v) = R^{-1} + \int_0^v dv'(v-v')P(v')$, from which $P(v) = Q''(v)$. Thus Eq. (13) becomes

$$Q(v)Q''(v) = R^{-1}P_0(v). \tag{14}$$

The boundary conditions are $Q(0) = R^{-1}$ and $Q'(0) = 0$. The auxiliary function $Q(v)$ gives a comprehensive description of the steady state. Calculation of important quantities such as the flux J requires knowledge of $G(v)$, the car velocity distribution. This quantity satisfies $1 = \int dv\, G(v)$ and $J = \int dv\, v\, G(v)$. Following an involved calculation that requires solution of a higher order velocity distribution [23], the car velocity distribution is derived explicitly in terms of $Q(v)$

$$G(v) = P(v)\left[1 + R\int_v^\infty dw\, P_0(w)\int_v^w \frac{du}{[RQ(u)]^2}\right]. \tag{15}$$

Hence, for arbitrary intrinsic velocity distributions, the entire steady state problem is reduced to the nonlinear second order differential equation (14). Given $Q(v)$, steady state distributions such as $P(v)$ and $G(v)$ can be calculated using the explicit formulas above.

Except for a few special cases, one can not solve the differential equation (14) analytically. Nevertheless, the formal solution above can be used to evaluate generic features of the flow. The dimensionless collision number R is extremely useful. For low collision numbers, a perturbation solution in powers of R can be constructed, as the steady state differs weakly from the initial state. For high collision numbers, a boundary layer analysis is possible as sufficiently small velocities are not affected by collisions. These two limits are quantitatively analyzed as follows.

3.1 Low Collision Numbers

The flow characteristics in the collision-controlled regime, $R \ll 1$, can be analyzed systematically as a perturbation series in R. For example, the cluster velocity distribution and the car velocity distribution read

$$\begin{aligned} P(v) &\cong P_0(v)\left[1 - R\int_0^v dv'(v-v')P_0(v')\right], \\ G(v) &\cong P_0(v)\left[1 + R\int_0^\infty dv'(v'-v)P_0(v')\right]. \end{aligned} \tag{16}$$

Consequently, average quantities such as the flux and the average cluster size vary linearly in R in this free flow regime, $J = J_0 - \text{const} \times R$, and $\langle m \rangle = 1 + \text{const} \times R$. The proportionality constant in the case of the flux equals the variance in the initial velocity distribution, indicating that the larger the initial velocity fluctuations, the larger the reduction in the flux.

Therefore, weakly interacting "laminar" flows arise in the $R \to 0$ limit. Technically, the steady state remains close to the initial state and a perturbation series in the collision number is possible. Here, the assumptions made in our theory are justified, as the cluster sizes are small, and at the leading order, a simplified model were all cars in the cluster can pass coincides with a more realistic model where only the first few cars may pass. In fact, a basic prediction of the model, namely linear growth of the average cluster size with the flux is consistent with empirical data, obtained from observations of traffic flows in a secondary rural road in Los Alamos, New Mexico [23].

3.2 High Collision Numbers

The limit of high collision numbers corresponds to dense, congested flows where large clusters form. The analysis in this passing-controlled regime is more subtle since the condition $R\int_0^v dv'(v-v')P_0(v') \ll 1$ is satisfied only for small velocities. No matter how large R is, sufficiently slow cars are not affected by collisions, and $P(v)$ is still given by Eq. (16) when $v \ll v^*$. The threshold velocity $v^* \equiv v^*(R)$ is estimated from $R\int_0^{v^*} dv(v^*-v)P_0(v) \sim 1$.

In the limit $R \to \infty$ limit, statistics of the slowest cars dominate the flow. Again, it is useful to consider intrinsic distributions with an algebraic small velocity form (4). For such distributions, the threshold velocity decreases with growing R according to

$$v^* \sim R^{-\frac{1}{\mu+2}}. \tag{17}$$

One can show that the flux is proportional to this velocity $J \sim v^*$. For $v \gg v^*$, the collision integral in Eq. (13) dominates over the constant factor and $RP(v)\int_0^v dv'(v-v')P(v') \sim v^\mu$. Anticipating an algebraic behavior for the cluster velocity distribution, $P(v) \sim R^\sigma v^\delta$ when $v \gg v^*$, gives different answers dependent on whether the cutoff exponent μ is positive or negative. The leading behavior for $v \gg v^*$ can be summarized as follows

$$P(v) \sim \begin{cases} (v^*)^\mu (v/v^*)^{\mu-1} & \mu < 0; \\ (v/v^*)^{-1}[\ln(v/v^*)]^{-\frac{1}{2}} & \mu = 0; \\ (v^*)^\mu (v/v^*)^{\frac{\mu}{2}-1} & \mu > 0. \end{cases} \tag{18}$$

On the other hand, $P(v) \cong P_0(v)$ for $v \ll v^*$. This shows that the velocity distribution develops a boundary layer structure, the size of which vanishes in the infinite collision number limit. Inside the boundary layer, the velocity distribution is only marginally lower than its initial values, while the bulk of the velocities are strongly suppressed. Similar to the threshold velocity

v^*, macroscopic characteristics of the flow depend algebraically on R. For example, the average cluster size is

$$\langle m\rangle \sim \begin{cases} R^{(\mu+1)/(\mu+2)} & \mu < 0; \\ (R/\ln R)^{1/2} & \mu = 0; \\ R^{1/2} & \mu > 0. \end{cases} \tag{19}$$

Two distinct regimes of behavior emerge. For $\mu > 0$, car-cluster collisions dominate while for $\mu < 0$ cluster-cluster collisions dominate. Interestingly, in the cluster-cluster dominated regime, $\langle m\rangle \sim R^\alpha$ with the scaling exponent $\alpha = (\mu+1)/(\mu+2)$ as in the no-passing case (3). Thus in the passing case the cutoff exponent μ also plays an important role in characterizing the behavior. Moreover, the steady state behavior is much richer than that found for the clustering kinetics.

Despite the simplifying assumptions, the model results in realistic behavior. The overall picture is both familiar and intuitive: due to the presence of slower cars, clusters form and the overall flux is reduced. For heavy traffic, the characteristics of the flow are solely determined by the distribution of slow cars. A single dimensionless parameter, the collision number R, ultimately determines the nature of the steady state.

4 The Maxwell model

While a comprehensive analysis of the steady state velocity distributions is possible using the approximate kinetic theory (12), other important questions such as the relaxation toward the steady state and the nature of the cluster size distribution [43] remain unanswered. To address these issues we consider a further approximation where the collision rate is taken to be uniform [25,26]. This approximation, known as the Maxwell model, is very useful in kinetic theory [44] and it has been recently applied to granular gases as well [45–47]. In our case, it allows for a complete exact solution of the time dependent behavior, and additionally, it leads to closed evolution equations for the cluster-size distribution.

4.1 Relaxation

In the Maxwell approximation, the collision rate $v - v'$ in the Boltzmann equation (12) is replaced by a constant factor which we set equal to unity. The corresponding rate equation for the cluster velocity distribution reads

$$\frac{\partial P(v,t)}{\partial t} = R^{-1}\left[P_0(v) - P(v,t)\right] - P(v,t)\int_0^v dv' P(v',t). \tag{20}$$

Again, the analysis is performed via a properly defined auxiliary function, $Q(v,t) = \int_0^v dv' P(v',t)$. The constant collision rate results in simpler differential equations, that are only first order in the velocity. The analog of

Eq. (14) is the integrable steady state equation $Q(v)Q'(v) = R^{-1}P_0(v)$. The resulting steady state properties are governed by R, with a boundary layer structure in the large R regime. The quantitative characteristics are somewhat different and for example the threshold velocity decays with R according to $v^* \sim R^{-1/(\mu+1)}$ rather than Eq. (17). If, however, the collision rate is properly chosen, namely set equal to $\langle v \rangle$ rather than unity, we recover Eq. (17).

Furthermore, the complete time dependence can be obtained analytically by integrating the partial differential equation $Q_t = R^{-1}Q_v - QQ_v$. In general, the relaxation is exponential $P(v,t) - P(v,t=\infty) \sim f(v)e^{-t/\tau(v)}$, with $\tau(v) = R\,[1 + 2RI_0(v)]^{-1/2}$ where $I_0(v) = \int_0^v dv' P_0(v')$. The relaxation time depends on the velocity and the collision number according to

$$\tau(v) \sim \begin{cases} R & v \ll v^*; \\ [R/I_0(v)]^{1/2} & v \gg v^*. \end{cases} \tag{21}$$

While small velocities are governed by practically fixed relaxation times, large velocities are characterized by velocity dependent decay rates. Furthermore, a large range of relaxation scales exists, $R^{1/2} < \tau < R$, with larger scales corresponding to smaller velocities. Further analysis shows that the same relaxation times underlie the car velocity distribution. We expect that while the predictions of the Maxwell model are only approximate, it correctly predicts the existence of a spectrum of relaxation time scales, and that the qualitative nature of the time dependent behavior generally holds.

4.2 The size distribution

The size distribution obeys closed evolution equations in the Maxwell model and can be solved exactly [25,26]. It can also be used to address the nature of the passing mechanism. To demonstrate this we consider the model where only the next-to-leading car in the cluster may pass and resume driving with its intrinsic velocity. From numerical simulations of this model, we find two distinct phases. In the laminar regime, clusters are generally small, specifically the cluster size distribution is exponentially suppressed for sufficiently large sizes. When the collision number exceeds a certain threshold, an infinite cluster is formed, i.e., a finite fraction of the cars in the system are in the cluster behind the slowest car. Furthermore, in this jammed phase the size distribution of finite clusters has a fat tail close to a power-law, $P_m \sim m^{-\tau}$, with $\tau \approx 2$.

In the Maxwell model framework, the cluster size distribution $P_m(t)$ obeys a closed system of rate equations

$$\frac{dP_m}{dt} = R^{-1}[P_{m+1} - P_m] - c\,P_m + \frac{1}{2}\sum_{i+j=m} P_iP_j, \qquad m \geq 2, \tag{22}$$

$$\frac{dP_1}{dt} = R^{-1}[P_2 - P_1 + c] - c\,P_1. \tag{23}$$

These equations were derived by enumerating all possible ways in which clusters evolve. For instance, consider Eq. (23). Collisions reduce the density of single cars, and the collision rate is clearly equal to c, as it is velocity-independent. The escape term in Eq. (23) is understood by observing that the rate of return of single cars into the system is $2P_2+\sum_{j\geq 3} P_j = P_2-P_1+c$. Here P_2 is singled out since passing transforms it into two single cars while an escape from larger clusters produces only one freely moving car.

Similar equations were previously studied in the context of aggregation-fragmentation processes [48–50]. Utilizing the approach of Ref. [48] we find that a phase transition occurs at $R_c = 1$ [26]. For large m, the steady state size distribution is

$$P_m \sim \begin{cases} m^{-3/2}\left[1-(1-R)^2\right]^m & R < 1, \\ m^{-5/2} & R \geq 1. \end{cases} \tag{24}$$

Hence in the laminar regime, the size distribution decays exponentially in the large size limit. In the congested phase, the size distribution has a power law tail, and in addition there is an infinite cluster that contains the following finite fraction of cars in the system:

$$I = \begin{cases} 0, & R < 1; \\ 1-R^{-1}, & R > 1. \end{cases} \tag{25}$$

Interestingly, this phase transition is similar to phase transitions in driven diffusive systems *without* passing [28–32]. Furthermore, The formation of an infinite cluster is reminiscent of Bose-Einstein condensation [30,49].

5 Discussion

The most important question raised by the above results concerns the validity of the "mean-field" Boltzmann equation (12). Although passing is a stochastic mixing mechanism that diminishes correlations between the velocities and the positions of the cars, such correlations do exist, and it will be interesting to determine whether quantitative predictions such as the scaling behaviors (17) and (19) are altered by spatial correlations. Similarly, the collision term in Eq.(22) is written in a mean-field spirit and that may be the reason for the discrepancy between the theoretical prediction $\tau = 5/2$ and the numerically observed value $\tau \approx 2$ of the decay exponent $P_m \sim m^{-\tau}$.

The primary feature of our model is *quenched disorder*, which manifests itself in the random assignment of intrinsic velocities. Road conditions (construction zones, intermittent passing zones, turns, hills, etc.) present another source of quenched randomness in actual roads [51], which is ignored in our model. Quenched disorder significantly affects characteristics of many-particle systems, especially in low spatial dimensions [52]. We have seen that this general conclusion clearly applies to our one-dimensional traffic model. Little is known analytically on the influence of the spatial disorder.

Finally, one may modify the passing rule so that when a car overtakes a slow car, it acquires a new velocity drawn from the distribution $P_0(v)$ rather than a pre-assigned velocity [53]. This elementary zero-memory model remains highly non-trivial even in the collision-controlled limit $R \to 0$ where clustering can be disregarded. The fate of the system is again determined by the behavior of the intrinsic velocity distribution near its lower cutoff. If $P_0(v)$ vanishes in this limit, the system reaches a steady state, otherwise, the system evolves indefinitely. Specifically, for intrinsic distributions with an algebraic small velocity tail (4) the long-time asymptotics of the average velocity reads

$$\langle v(t)\rangle \sim \begin{cases} \text{const} & \mu > 0; \\ (\ln t)^{-1} & \mu = 0; \\ t^{\mu} & -1 < \mu < 0. \end{cases} \tag{26}$$

These results were derived in a simplified Boltzmann framework. In particular, the most interesting behavior in the evolving regime was obtained by assuming that as $t \to \infty$, cars can be divided into two groups, the small group of "active" cars which move with velocities $v \sim 1$ and the vast majority of "creeping" cars that hardly move at all. We then ignored collisions between creeping cars (since their relative velocity is very small) and collisions between active cars (since their density is small). Thence, the velocity distribution of active cars obeys a linear Boltzmann-Lorentz equation which was solved to give (26). Comparison with results of molecular dynamics simulations suggests that the mean-field theory description is asymptotically exact. It will be interesting to confirm this result rigorously.

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