Finite Size Fluctuations in Interacting Particle Systems

E. Ben-Naim*
Theoretical Division and Center for Nonlinear Studies,
Los Alamos National Laboratory, Los Alamos, New Mexico, 87545

P. L. Krapivsky†
Center for Polymer Studies and Department of Physics, Boston University, Boston, MA, 02215

Fluctuations may govern the fate of an interacting particle system even on the mean-field level. This is demonstrated via a three species cyclic trapping reaction with a large, yet finite number of particles, where the final number of particles N_f scales logarithmically with the system size N, $N_f \sim \ln N$. Statistical fluctuations, that become significant as the number of particles diminishes, are responsible for this behavior. This phenomenon underlies a broad range of interacting particle systems including in particular multispecies annihilation processes.

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I. INTRODUCTION

Balls-in-boxes (urn) models provide a handy laboratory for studying conceptual issues. The celebrated Erenfest model [1–3], for instance, has led to the reconciliation of the reversibility and the recurrence paradoxes with Boltzmann's H theorem. Recent examples include urn models of aging [4–6] and of discretized quantum gravity [7]. Urn models have been studied extensively in probability theory [8–10] and have found applications ranging from biology [11] to computer science [12, 13].

Perhaps the most well-known manifestation of the role of fluctuations in stochastic processes is the Eggenberger-Pólya urn model [14]. One starts with two marbles of different colors, draws a marble randomly and puts it back together with another marble of the same color. When the process is repeated indefinitely, the fraction of marbles of a given color saturates at some limiting value. The corresponding limit is a random variable that is uniformly distributed between zero and one. Thus, initial fluctuations are locked in, a striking example of the lack of self-averaging.

Inspired by this example, our goal is to quantify finite size fluctuations in interacting particle systems using urn models. Finite-size corrections are important because they govern for example how a system converges to the thermodynamic limit, yet they remain largely unresolved even in elementary stochastic processes [15–20].

We study the role of fluctuations using a three color cyclic urn model. Initially, the urn contains three different types of balls. Then, two balls are picked randomly. If they are dissimilar, following a cyclic rule, one of the balls is returned to the urn and the other is removed from the system. This urn model is different from the Eggenberger-Pólya type models in two ways. First, the number of balls is *decreasing* rather than increasing. Second, our model is *nonlinear* because picking two balls rather than one implies that different type balls interact with each other.

We define the state of the system when one of the species is depleted to be the final state. Starting from the natural initial conditions where there are N balls of each type, we ask: "How many balls are there in the final state?". Our main result is that the average number of balls in the final state N_f scales logarithmically with the system size N.

Statistical fluctuations are ultimately responsible for this behavior. As long as the system contains enough particles (balls), the average number of particles faithfully characterizes the state of system. However, as particles deplete, the uncertainty with respect to how many particles remain grows and moreover, it governs how many particles are finally left.

This phenomenon and the mechanism underlying it are generic to interacting particle systems with a decreasing number of particles. We demonstrate this by considering multi-species annihilation processes with q species. In the three species model, there is again a logarithmic enhancement of the variance in the number of particles over the average number of particles, thereby leading to a logarithmic growth law. In general, there is an algebraic enhancement as long as the number of species is small enough, and consequently, an algebraic growth law. Otherwise, when the number of different species is large enough, statistical fluctuations are negligible and the number of remaining particles is of order one. In the most general case when p balls are drawn from the urn, the critical number of species is $q_c = 2p - 1$.

The rest of the paper is organized as follows. In Sec. II, we give a nontechnical presentation of the cyclic trapping model and highlight the basic result. We then analyze the model in detail and obtain the number density fluctuations by employing the $1/N$ expansion (Sec. III). The q-species annihilation model is treated in Sec. IV. We conclude with a few open questions (Sec. V).

*Electronic address: ebn@lanl.gov
†Electronic address: paulk@bu.edu

II. CYCLIC TRAPPING REACTIONS

Let us first define the model. Initially, the urn contains balls of three different types, denoted by A, B, and C. The balls interact via a cyclic trapping reaction. Two balls are taken randomly out of the box. If they are different, then one of the balls is returned to the urn according to the cyclic rule

$$A + B \to B, \qquad B + C \to C, \qquad C + A \to A, \tag{1}$$

and the second ball is discarded. This elemental step is repeated until one of the species becomes extinct. This reaction scheme is reminiscent of the Lotka-Volterra cyclic food chain (or rock-paper-scissors) model, widely used in ecology and game theory [11, 21–24].

The state of the system is fully specified by the number of particles of each type in the urn: n, m, and l, corresponding to particles of type A, B, and C. The dynamics is clearly mean-field: every dissimilar pair of particles is equally likely to interact. Therefore, the transition $(n, m, l) \to (n-1, m, l)$ occurs with probability $nm/(nm+ml+ln)$ and similarly for the two other transitions.

We start with the initial condition $n = m = l = N$ and stop the process when one species becomes extinct. Surprisingly, the number of balls in the final state scales logarithmically with the system size:

$$N_f \sim \ln N. \tag{2}$$

Numerical simulations are consistent with this behavior (Fig. 1). We verified numerically that the scale $\ln N$ fully characterizes the final number of particles. The distribution of the final number of particles approaches a (nontrivial) limiting distribution when the final number of particles is normalized by $\ln N$. Thus, the final number of particles in a non-self-averaging quantity.

The problem is essentially combinatorial, and in principle, it can be addressed by weighing all possible histories with the appropriate transition probabilities [25]. It proves fruitful, however, to treat the process dynamically. Choosing the rate nm/N for the transition $(n, m, l) \to (n, m-1, l)$ is consistent with the above microscopic rules. Moreover, it leads to N-independent dynamics in the thermodynamic limit.

The number densities $a = \langle n\rangle/N$, $b = \langle m\rangle/M$, and $c = \langle l\rangle/N$ evolve according to the rate equations

$$\frac{da}{dt} = -ab, \qquad \frac{db}{dt} = -bc, \qquad \frac{dc}{dt} = -ca. \tag{3}$$

Since initially $a(0) = b(0) = c(0) = 1$, the number densities remain equal throughout the entire process $a(t) = b(t) = c(t) = \rho(t)$ with

$$\rho(t) = (1+t)^{-1}. \tag{4}$$

Naively assuming that throughout the process, fluctuations in the number density are much smaller than the mean leads to the conclusion that the final number of particles is of the order unity, $N_f \sim \mathcal{O}(1)$. The corresponding terminal time scales linearly with the system size, $t_f \sim N$. Below, we show that this assumption does not hold when the total number of particles becomes sufficiently small.

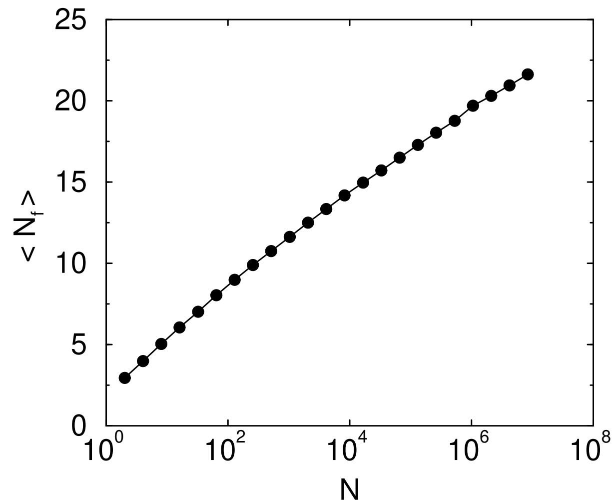

FIG. 1: The average total number of particles in the final state as a function of the system size. The data represents an average over 10^5 realizations of the cyclic trapping reaction process (1).

The logarithmic growth in the number of particles can be deduced from the fluctuations in the number density. In the thermodynamic limit, we expect that to leading order in N, both the total number of particles and the variance in the number of particles are proportional to the system size

$$\begin{aligned} \langle n\rangle &\simeq N\rho, \\ \langle n^2\rangle - \langle n\rangle^2 &\simeq N\sigma^2. \end{aligned} \tag{5}$$

We term σ^2 the intrinsic variance. In Sec. III, we shall utilize the van Kampen $1/N$ expansion [26, 27] to show that asymptotically, the ratio between the intrinsic variance and the density grows logarithmically with time

$$\frac{\sigma^2}{\rho} \sim \ln t. \tag{6}$$

Thus, fluctuations eventually become larger then the density. Of course, when they are comparable with the density, extinction is possible. Hence, the number density (4) characterizes the particle number only up to a time scale t_f obtained from the validity criterion $N\rho(t_f) \sim \sqrt{N\sigma^2(t_f)}$. The terminal time is therefore

$$t_f \sim N(\ln N)^{-1}. \tag{7}$$

Using $N_f \sim N\rho(t_f)$ we arrive at our main result (2). Note that $\ln N$ is the leading contribution. The subleading contribution $\ln(\ln N)$ corresponding to nested logarithms is tacitly ignored.

III. PARTICLE NUMBER FLUCTUATIONS

Fluctuations in the particle number are studied by expanding the master equation in inverse powers of N and keeping only the leading order terms (large-N expansion) [26]. The probability $P(n,m,l;t)$ that the particle numbers are n, m, and l at time t obeys the master equation

$$\frac{d}{dt}P(n,m,l) = (\mathcal{L}_{AB} + \mathcal{L}_{BC} + \mathcal{L}_{CA})\, P(n,m,l) \qquad (8)$$

with the initial condition $P_0(n,m,l) = \delta_{n,N}\delta_{m,N}\delta_{l,N}$. The operator $\mathcal{L}_{AB}$ is

$$\mathcal{L}_{AB}P(n,m,l) = N^{-1}(\Delta_A - 1)\,[nmP(n,m,l)]\,; \qquad (9)$$

the operators $\mathcal{L}_{BC}$ and $\mathcal{L}_{CA}$ are defined via similar formulas. The difference operator Δ raises the respective variable by one, e.g.

$$\Delta_A\, f(n,m,l) = f(n+1,m,l). \qquad (10)$$

Since in the thermodynamic limit, averages as well as variances grow linearly with the system size as in Eq. (5), we introduce the transformation $P(n,m,l) \to F(\alpha,\beta,\gamma)$ with

$$n = Na + N^{1/2}\alpha, \quad m = Nb + N^{1/2}\beta, \quad l = Nc + N^{1/2}\gamma.$$

The intensive (random) variables α, β, and γ are N-independent. These variables simply characterize fluctuations in the respective particle numbers.

To find out how the distribution $F(\alpha,\beta,\gamma)$ evolves with time, we write

$$F_t = (\mathcal{M}_{AB} + \mathcal{M}_{BC} + \mathcal{M}_{CA})F. \qquad (11)$$

It suffices to compute the evolution operator $\mathcal{M}_{AB}F$; the two other operators are obtained by cyclic transposition. To obtain the evolution operators, we replace the distribution P by F in (8) and convert difference equations into differential ones by expanding difference operators and keeping up to second order terms, e.g., $\Delta_A \to 1 + \partial_n + \frac{1}{2}\partial_{nn}$. Similarly, we replace derivatives with respect to n with derivatives with respect to α using $\partial_n = N^{-1/2}\partial_\alpha$. The time derivative becomes $\partial_t - N^{1/2}\dot{a}\partial_\alpha - N^{1/2}\dot{b}\partial_\beta - N^{1/2}\dot{c}\partial_\gamma$ where the overdot denotes differentiation with respect to time. These transformations lead to

$$F_t - N^{1/2}(\dot{a}F_\alpha + \dot{b}F_\beta + \dot{c}F_\gamma) = (N^{-1}\partial_\alpha + \frac{1}{2}N^{-1/2}\partial_{\alpha\alpha})[(Na + N^{1/2}\alpha)(Nb + N^{1/2}\beta)F] + (\text{c.t.}) \qquad (12)$$

where (c.t.) denotes the two terms obtained by cyclic transposition of the displayed term on the right-hand side. This master equation contains terms of various orders in N. The order $N^{1/2}$ terms vanish because the densities satisfy the rate equations (3). The next leading order term gives the evolution operator

$$\mathcal{M}_{AB}F = \rho\partial_\alpha[(\alpha+\beta)F] + \frac{1}{2}\rho^2 F_{\alpha\alpha}. \qquad (13)$$

Explicitly, the Fokker-Planck equation (11) is

$$\begin{aligned} F_t &= \rho[\partial_\alpha(\alpha+\beta) + \partial_\beta(\beta+\gamma) + \partial_\gamma(\gamma+\alpha)]F \\ &+ \frac{1}{2}\rho^2(F_{\alpha\alpha} + F_{\beta\beta} + F_{\gamma\gamma}). \end{aligned} \qquad (14)$$

This Fokker-Planck equation is subject to the initial condition $F_0(\alpha,\beta,\gamma) = \delta(\alpha)\,\delta(\beta)\,\delta(\gamma)$.

Moments of the probability distribution $F(\alpha,\beta,\gamma)$ directly follow from (14). One simply multiplies this Fokker-Planck equation by the desired powers of α, β, and γ, and integrates (by parts) with respect to these three variables. Due to symmetry, there is essentially one first moment: $\langle\alpha\rangle$; two second moments: $\langle\alpha^2\rangle, \langle\alpha\beta\rangle$; three third moments: $\langle\alpha^3\rangle, \langle\alpha^2\beta\rangle, \langle\alpha\beta\gamma\rangle$; etc. The first moment satisfies $\frac{d}{dt}\langle\alpha\rangle = -2\rho\langle\alpha\rangle$ and since it vanishes initially, $\langle\alpha\rangle = 0$. The two second moments are coupled

$$\begin{aligned} \frac{d\langle\alpha^2\rangle}{dt} &= -2\rho\langle\alpha^2\rangle - 2\rho\langle\alpha\beta\rangle + \rho^2, \\ \frac{d\langle\alpha\beta\rangle}{dt} &= -\rho\langle\alpha^2\rangle - 3\rho\langle\alpha\beta\rangle. \end{aligned} \qquad (15)$$

These equations are inhomogeneous, so despite of the vanishing initial conditions $\langle\alpha^2\rangle = \langle\alpha\beta\rangle = 0$, the solutions are non-trivial.

Writing $U = \langle\alpha^2\rangle + 2\langle\alpha\beta\rangle$ and $V = \langle\alpha^2\rangle - \langle\alpha\beta\rangle$, we separate the above equations

$$\begin{aligned} \frac{dU}{dt} &= -4\rho U + \rho^2, \\ \frac{dV}{dt} &= -\rho V + \rho^2. \end{aligned} \qquad (16)$$

Using the number density (4), we obtain the explicit expressions

$$\begin{aligned} U &= \frac{1}{3}\left[(1+t)^{-1} - (1+t)^{-4}\right], \\ V &= (1+t)^{-1}\ln(1+t). \end{aligned} \qquad (17)$$

Physically, $U = \langle\alpha(\alpha+\beta+\gamma)\rangle$ quantifies the correlation between the single particle number n and the total particle number $n+m+l$, while $V=\langle\alpha(\alpha-\beta)\rangle$ quantifies the correlation between the particle number n and the number difference $n-m$. Intuitively, we expect that the quantity V is larger than U. For a sufficiently large system, it may be arbitrarily larger.

One of the two second moments is the intrinsic variance $\langle\alpha^2\rangle\equiv\sigma^2$; explicitly,

$$\frac{\sigma^2}{\rho}=\frac{2}{3}\left[\ln(1+t)+\frac{1}{6}-\frac{1}{6}(1+t)^{-3}\right]. \tag{18}$$

The other (normalized by the density) second moment quantifies cross-correlations between different species numbers

$$\frac{\langle\alpha\beta\rangle}{\rho}=-\frac{1}{3}\left[\ln(1+t)-\frac{1}{3}+\frac{1}{3}(1+t)^{-3}\right]. \tag{19}$$

The quantity $\langle\alpha\beta\rangle$ is always negative and therefore, fluctuations between different particle numbers are anti-correlated. Asymptotically, $\langle\alpha\beta\rangle\simeq-\sigma^2/2$ with

$$\sigma^2\simeq\frac{2}{3}\,t^{-1}\ln t. \tag{20}$$

Another important consequence of the structure of the Fokker-Planck equation is that the multivariate distribution $P(n,m,l)$ is Gaussian and fully characterized by the first and second order moments. This is the case because the first order derivatives in (14) have linear coefficients [26]. As a result, the individual particle number distribution is also Gaussian

$$P(n,t)\simeq\frac{1}{\sqrt{2\pi N\sigma^2}}\exp\left[-\frac{(n-N\rho)^2}{2N\sigma^2}\right]. \tag{21}$$

IV. MULTISPECIES ANNIHILATION

We have examined the question "how many particles remain in the final state?" in a number of other interacting particle systems where depletion or extinction occurs and found that generically, fluctuations play an important role. Using the same validity criterion, and utilizing the van Kampen's $1/N$ expansion, one can determine the final number of remaining particles.

We demonstrate this for multi-species annihilation processes. Initially, the urn contains q types of balls and the initial number of each species is equal to N. For instance, when $q=3$,

$$A+B\to 0,\qquad B+C\to 0,\qquad C+A\to 0. \tag{22}$$

This process, introduced by ben-Avraham and Redner, was studied primarily in low spatial dimensions via a number of numerical and analytical techniques, yet it is still not fully understood [27–29].

The parameter q is in principle integer. However, it is still sensible to treat it as a continuous variable in the range $2<q<\infty$. The q-species annihilation process can be reformulated as a two-species annihilation model by combining $q-1$ of the species into one group (A) and the remaining specie into a second group (B) [30]. The reaction scheme becomes $A+A\to 0$ and $A+B\to 0$. The ratio between the reaction rates of the two channels, $\frac{q-2}{q-1}$, is a continuous parameter that need not necessarily correspond to an integer q.

The transition rates are as in the cyclic trapping reaction: $(n,m,l,\ldots)\to(n-1,m-1,l,\ldots)$ occurs with rate nm/N. For symmetric initial conditions, the number density $\rho=a=b=c=\cdots$ satisfies

$$\frac{d\rho}{dt}=-(q-1)\rho^2, \tag{23}$$

and the initial condition $\rho(0)=1$. The concentration is therefore

$$\rho=[1+(q-1)t]^{-1}. \tag{24}$$

Fluctuations can be obtained following the same straightforward steps the led to the evolution equations for the moments and we merely highlight the derivation. For $q=3$, the probability distribution $P(n,m,l)$ evolves according to (8) with the operator $\mathcal{L}_{AB}$ defined via

$$\mathcal{L}_{AB}P=N^{-1}\left(\Delta_A\Delta_B-1\right)[nmP]. \tag{25}$$

The probability distribution $F(\alpha,\beta,\gamma)$ evolves according to (11) with the evolution operator now being

$$\mathcal{M}_{AB}F=\rho(\partial_\alpha+\partial_\beta)[(\alpha+\beta)F]+\frac{\rho^2}{2}\left(\partial_\alpha+\partial_\beta\right)^2F. \tag{26}$$

For arbitrary q, there are $q(q-1)/2$ such operators. Again, the first moment of F vanish; the second moments are coupled as follows

$$\begin{aligned}\frac{d\langle\alpha^2\rangle}{dt}&=-2(q-1)\rho\langle\alpha^2\rangle-2(q-1)\rho\langle\alpha\beta\rangle+(q-1)\rho^2,\\ \frac{d\langle\alpha\beta\rangle}{dt}&=-2\rho\langle\alpha^2\rangle-2(2q-3)\rho\langle\alpha\beta\rangle+\rho^2.\end{aligned} \tag{27}$$

In contrast with the cyclic trapping reaction, the cross-correlation initially grows, although asymptotically it is again negative.

Let $U=\langle\alpha^2\rangle+(q-1)\langle\alpha\beta\rangle$ and $V=\langle\alpha^2\rangle-\langle\alpha\beta\rangle$; the former quantity measures the correlation between n and the total particle number, the latter measures the correlation between n and $n-m$. To treat different values of q on the same footing, we rescale the time variable and introduce $\tau=(q-1)t$. The number density (23) becomes $\rho=(1+\tau)^{-1}$ and the rate equations for the quantities U and V are

$$\begin{aligned}\frac{dU}{d\tau}&=-4\rho U+2\rho^2,\\ \frac{dV}{d\tau}&=-2\frac{q-2}{q-1}\rho V+\frac{q-2}{q-1}\rho^2.\end{aligned} \tag{28}$$

The solution for the first quantity is therefore q-independent and apart from the numerical prefactor, as in the cyclic trapping model,

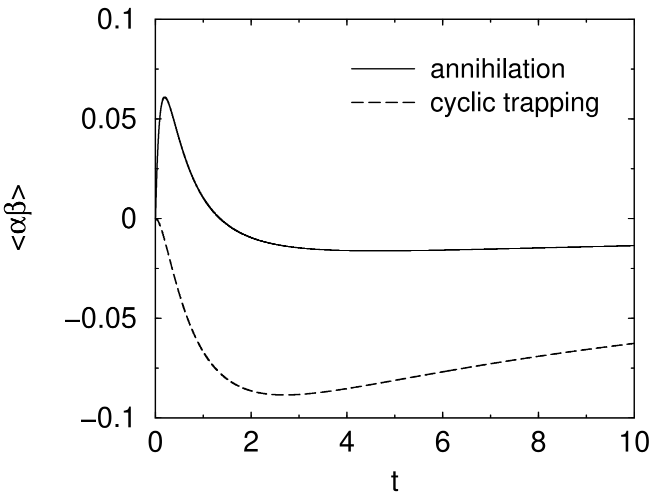

FIG. 2: The cross-correlation versus time for the cyclic trapping model and the three-species annihilation.

$U = \frac{2}{3}\left[(1+\tau)^{-1} - (1+\tau)^{-4}\right]$. Two different behaviors are found for the second quantity:

$$V = \begin{cases} \frac{q-2}{q-3}\left[(1+\tau)^{-1} - (1+\tau)^{-2\frac{q-2}{q-1}}\right] & q \neq 3 \\ \frac{1}{2}(1+\tau)^{-1}\ln(1+\tau) & q = 3 \end{cases} \tag{29}$$

Asymptotically, the cross-correlation is negative because $\langle\alpha\beta\rangle \simeq -\frac{1}{q-2}V$ and so generically, fluctuations between the numbers of different species are anti-correlated. Early on, the cross-correlation increases, but after a short transient it becomes negative (Fig. 2).

In the long time limit, $\sigma^2 \simeq (1-q^{-1})V$:

$$\frac{\sigma^2}{\rho} \sim \begin{cases} t^{\frac{3-q}{q-1}} & q < 3; \\ \ln t & q = 3; \\ \mathcal{O}(1) & q > 3. \end{cases} \tag{30}$$

Therefore, fluctuations are relevant asymptotically only when $q \leq 3$. Applying the criterion $\sqrt{N\sigma^2(t_f)} \sim N\rho(t_f)$ yields the final time $t_f(N)$ and consequently, the typical final number of particles

$$N_f \sim \begin{cases} N^{(3-q)/2} & q < 3; \\ \ln N & q = 3; \\ \mathcal{O}(1) & q > 3. \end{cases} \tag{31}$$

There is an algebraic growth in the fluctuations dominated regime $q < 3$. At the critical point $q_c = 3$, logarithmic growth occurs. Otherwise, the final number of particles saturates at a finite value. Still, the final number diverges, $N_f \sim (q-3)^{-1}$, in the vicinity of the critical point, $q \downarrow 3$. The saturation is illustrated in Fig. 3 using numerical simulation data for the $q = 4$ case.

The case $q = 2$ is special since there is a conservation law ($n - m$ is conserved) and therefore, $V = 0$. Consequently, $\sigma^2 \sim \rho \sim t^{-1}$. If a q-species aggregation, rather than annihilation process is considered, that is, when A and B interact the outcome is either $A + B \to A$ or $A + B \to B$ (both taken with equal probability), then this anomaly disappears [27, 28] and Eq. (31) holds for $q = 2$ as well.

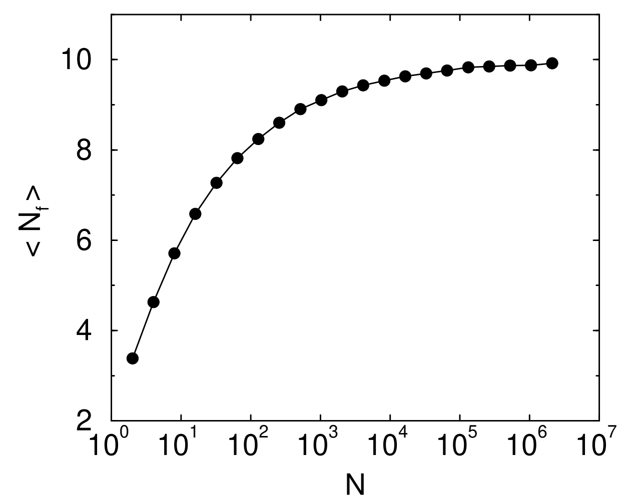

FIG. 3: The average total number of particles in the final state as a function of the system size for $q = 4$. The data represents an average over 10^5 realizations of the four-species annihilation process.

One may ask "why is the critical number of species equal to 3?". Mathematically, the answer is ultimately related to the smaller eigenvalue of the 2×2 matrix coupling the second moments. Yet given the simplicity of the multispecies urn model, a heuristic and more illuminating derivation may be possible after all. Finding such an argument is an interesting challenge.

In this context, we mention a generalization of the urn model from two-particle to the many-particle interactions. That is, instead of picking 2 particles, p particles are picked and if they are all of different species, all are removed from the system (this process is well defined for $p \leq q$). The rate equations for the second moments yield the critical number of species (see Appendix)

$$q_c(p) = 2p - 1. \tag{32}$$

Thus, for ternary interactions $q_c = 5$. The structure of the phase transition remains the same. There is an algebraic growth in the total number of remaining particles as a function of the system size when $q < q_c$, a logarithmic growth at the critical point $q = q_c$, and saturation above the critical point $q > q_c$. Last, we mention that a similar phase transition underlies two-species reaction processes of the type constructed from the q-species annihilation by separating species into two groups. In this case, although the transition depends on the reaction rates rather than the number of species, its structure remains the same.

V. CONCLUSION

In summary, we considered interacting particle systems undergoing depletion on the mean-field level. We showed that finite size fluctuations display a rich behavior. The behavior is rather generic and applies to a wide class of stochastic processes with a decreasing number of particles. Typically, there is a phase transition as a function of the number of species or the reaction rates. In one region of parameter space, the final number of particles grows algebraically, and in the other, it saturates at a finite value. The critical case is marked by a logarithmic growth. We conclude that the final number of particles as a function of system size provides a practical probe of statistical fluctuations.

The findings in the cyclic trapping model have a neat game theoretic implication. In a rock-paper-scissor game involving fixed strategy players and loser-is-out rules, the game ends when all remaining players have the same strategy. If players pair randomly, then the ultimate number of winners scales logarithmically with the total number of players. Intuitively, we expect that a similar law emerges for tournaments with simultaneous play.

Several questions arise naturally. Can one explain the critical number of species using heuristic arguments? Statistical properties of the final state and how the system approaches it are interesting as well. For example, what is the number distribution of remaining particles? How different are statistical properties of the system at the very end of the process when only a single species remains? We observed that the convergence to the asymptotic behavior is much faster when the first extinction occurs compared with the very end state when only a single species remains. Last, an interesting question involves extremal characterization [19, 20]: What is the probability that one of the species is always the most or least numerous?

We studied systems undergoing depletion. However, there are processes in which depletion is possible but not certain, for example, infection processes [21]. It will be interesting to investigate finite size fluctuations in this related class of interacting particle systems.

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APPENDIX A: p-PARTICLE ANNIHILATION

For p-particle annihilation, the density evolves according to

$$\frac{d}{dt}\rho = -\binom{q-1}{p-1}\rho^p. \tag{A1}$$

The evolution operators for the Fokker-Planck equation for F are straightforward generalizations of Eq. (26). For example, for the ternary ($p = 3$) annihilation process $A + B + C \to 0$,

$$\mathcal{M}_{ABC}F = \rho^2(\partial_\alpha + \partial_\beta + \partial_\gamma)[(\alpha+\beta+\gamma)F] + \frac{\rho^3}{2}\left(\partial_\alpha + \partial_\beta + \partial_\gamma\right)^2 F. \tag{A2}$$

The second moments are coupled as follows

$$\frac{d}{dt}\begin{pmatrix}\langle\alpha^2\rangle\\ \langle\alpha\beta\rangle\end{pmatrix} = -2\rho^{p-1}\begin{pmatrix}\binom{q-1}{p-1} & (p-1)\binom{q-1}{p-1}\\ \binom{q-2}{p-2} & (p-1)\binom{q-2}{p-2} + p\binom{q-2}{p-1}\end{pmatrix}\begin{pmatrix}\langle\alpha^2\rangle\\ \langle\alpha\beta\rangle\end{pmatrix} + \rho^p\begin{pmatrix}\binom{q-1}{p-1}\\ \binom{q-2}{p-2}\end{pmatrix}. \tag{A3}$$

Introducing the time variable $\tau = (p-1)\binom{q-1}{p-1}t$, the density is simply $\rho = (1+\tau)^{-1/(p-1)}$. The quantity $U = \langle\alpha^2\rangle + (q-1)\langle\alpha\beta\rangle$ satisfies $\frac{d}{d\tau}U + \frac{2p}{p-1}\rho^{p-1}U = \frac{p}{p-1}\rho^p$ and the solution is again q-independent

$$U(\tau) = \frac{p}{2p-1}\left[(1+\tau)^{-\frac{1}{p-1}} - (1+\tau)^{-\frac{2p}{p-1}}\right]. \tag{A4}$$

The quantity $V = \langle\alpha^2\rangle - \langle\alpha\beta\rangle$ satisfies $\frac{d}{d\tau}V + \frac{2(q-p)}{(p-1)(q-1)}\rho^{p-1}V = \frac{q-p}{(p-1)(q-1)}\rho^p$. The solution reads

$$V(\tau) = \begin{cases}\frac{q-p}{q-(2p-1)}\left[(1+\tau)^{-\frac{1}{p-1}} - (1+\tau)^{-\frac{2(q-p)}{(p-1)(q-1)}}\right] & q \neq 2p-1;\\ \frac{1}{2(p-1)}(1+\tau)^{-\frac{1}{p-1}}\ln(1+\tau) & q = 2p-1.\end{cases} \tag{A5}$$

Interestingly, in the fluctuation dominated regime, $q < 2p-1$, the exponent governing the terminal time is p-independent, $t_f \sim N^{\frac{q-1}{2}}$. The final number of particles is

$$N_f \sim \begin{cases} N^{\frac{2p-1-q}{2(p-1)}} & q < 2p-1;\\ \ln N & q = 2p-1;\\ \mathcal{O}(1) & q > 2p-1.\end{cases} \tag{A6}$$